We have been working on several topics that were planned in the description of this OTKA grant. The outcome is almost 50 publications during this period, and the number is going to be more 50 than when the papers we work on currently will be published.

Géza Tóth, together with István Kovács, proved that every closed, centrally symmetric convex polygon P is cover-decomposable, meaning that if the plane is covered sufficiently many times by translates of P, then the covering can be decomposed into two coverings. They also proved a stronger and more general statement that for any fixed k, if a planar set S is covered sufficiently many times by translates of P, then the covering can be decomposed into k coverings of S. Moreover, any infinite-fold covering of S can be decomposed into infinitely many coverings of S.

A thrackle is a graph drawn in the plane such that every two edges have either a common endpoint, or they cross. According to the 40 years old thrackle conjecture, a thrackle can have at most as many edges as vertices. Together with Pach and Radoičić, Tóth has investigated a modification of the problem, where touching edges are also allowed. The problem has several versions, (1) crossings are not allowed, only touchings, (2) both crossings and touchings are allowed, (3) the edges are x-monotone curves. In the case of 1 and 3 they proved tight bounds on the maximum number of edges, in 2 they still have a gap.

With Dániel Gerbner they showed that any set of n points in the plane can be separated by $O(n \log \log n / \log n)$ convex sets, and for some point sets $\Omega(n/ \log n)$ convex sets are necessary. Géza Tóth, Together with Z Lángi, M Naszódi, J Pach, and G Tardos, generalized this result in several directions, in particular, in arbitrary dimensions, arbitrary many faulty points, different types of questions, and in some other directions motivated by similar results in search theory. There are still many open problems, most of our results are far from optimal. We plan to continue our work on this group of problems.

A simple topological graph is a graph drawn in the plane so that any pair of edges have at most one point in common, which is either an endpoint or a proper crossing. It is called saturated if no further edge can without violating this condition. With Jan Kynčl, János Pach and Rados Radoičić, Géza Tóth constructed saturated simple topological graphs with n vertices and linear $(17.5n)$ number of edges. For every $k > 1$, they we gave similar constructions for k-simple topological graphs, that is, for graphs drawn in the plane so that any two edges have at most k points in common.

They also showed, that in any k-simple topological graph, any two independent vertices can be connected by a curve that crosses each of the original edges at most $2k$ times and this bound of $2k$ cannot be improved.

With E Ackerman, J Pach, R. Pinchasi and R Radoičić, G Tóth showed that the lines of every arrangement of n lines in the plane can be colored with $O(\sqrt{n/ \log n})$ colors such that no face of the arrangement is monochromatic. This improves a bound of Bose et al. Any further improvement on this bound would also improve the best known lower bound on the following problem of

Erdős: estimate the maximum number of points in general position within a set of n points containing no four collinear points.

Concerning lattice polytope questions, the PI has answered a question of V I Arnold on the number, $N(d, V)$, of (equivalence classes) of lattice polytopes in R^d of volume exactly V, by giving a lower bound on $\log N(d, V)$. An upper bound of the same order of magnitude has been known for almost twenty years. A second proof of the same lower bound, with interesting consequences is given in a joint work of the PI with Liping Yuan.

The PI has also proved that if a triangle contains a convex lattice chain of length n, then its area is at least $n^2(n-1)/8$, and characterized the case of equality for all large enough n (joint work with E Roldan Pensado). Significant new results were reached in combinatorial convexity: a Tverberg type theorem for coloured classes of hyperplanes, where we show that a certain homogeneous selection is always possible (by I Bárány and J Pach). New bounds were given on various Carathéodory numbers using topological and geometric methods (by I Bárány and R Karasev).

Some further results in the directions planned in the original proposal: the PI has shown (joint work with J-F Merckart and M Reitzner) that in a random sample of n points from a plane convex body there is always a pair which is the edge of many (namely $O(n/\log n)$) empty triangles. He proved further that a typical boundary point p of a typical 3-dimensional convex body has the following property: sections parallel to the tangent plane at p form a dense set in the space of shapes of 2-dimensional convex bodies (typical is meant both times in Baire category sense). This is joint work with R Schneider. The PI has also worked on an old question of Erdős on how many times the same distance can occur on a convex curve in the plane (joint work with E Roldan Pensado), and has made significant progress in this direction. Significant new results were reached in the geometry of Alexandrov spaces: we showed that every point is critical in the sense that it is a farthest point from some other point of the space (Joint paper with Itoh, Vilcu, and Zamfirescu). This is a surprising result and its proof uses methods from convex geometry, topology, and analysis.

The PI has shown (joint work with V Grinberg) that for every k, every finite sequence of real numbers in $[0, 1]$ can be split into k blocks (that is, subsequences formed by consecutive elements) so that the sum of the elements in the blocks differ by at most one. The PI has also worked (with J Castro) on a Helly type problem and showed that if we have an odd number of at most unit vectors (in any given norm) in the plane such that sum of any three has norm at least one, then the total sum of the vectors has norm at least one.

With J Matoušek and A Pór the PI established the order of magnitude of a geometric Ramsey number, which is a significant new result. The proof is based on the d-dimensional extension of the following result: if a plane curve has at most 3 points in common with every line, then it can be split into (at most) four convex subcurves. The PI has investigated the question:

what 3-dimensional convex bodies are fixed by what planar frames (joint works with T Zamfirescu).

Further results have been proved on curvature properties of convex bodies of constant width (PI and R Schneider), on the average number of affine diameters of a convex body K in R^d, containing a fixed point (PI, D. Hug, R. Schneider), on small subset sums (G. Ambrus, PI, V. Grinberg).

One of the most interesting new result from this period is an extension the famous Erdős-Szekeres theorem to finite families of lines in the plane. This is a joint work of I Bárány, E Roldan-Pensado and G Tóth. While the Erdős-Szekeres number for points is known to be between roughly 2^n and 4^n, and the lower bound is conjectured to be the truth, we showed that, quite surprisingly, the corresponding number for lines is very close to 4^n. We plan to work further on this exciting new phenomenon.