# Final report on the OTKA grant K83267

## Subject and motivation of the research

Recent advances in lattice QCD make it possible to measure relevant physical quantities at realistic, physical quark masses. This includes the measurement of the nuclear force between nucleons by the HAL QCD collaboration [1, 2, 3]. The HAL QCD method is based on the Nambu-Bethe-Salpeter (NBS) wave function defined by

$$\psi_{\mathbf{k}}^{\mathrm{NBS}}(\mathbf{x}) = \langle 0|N(\mathbf{0},0)N(\mathbf{x},0)|\mathrm{NN};\mathbf{k}\rangle^{\mathrm{in}}, \tag{1}$$

where $\langle 0|$ is the QCD vacuum state, $|\mathrm{NN};\mathbf{k}\rangle^{\mathrm{in}}$ is a 2-nucleon scattering state in the centre-of-mass (COM) frame with nucleon momenta $\mathbf{k}$ and $-\mathbf{k}$ and $N(\mathbf{x},t)$ is a local nucleon field operator. Both the nucleon field operators and the 2-nucleon state depend on additional quantum numbers (spin, isospin, etc.), which are suppressed in the above formula for simplicity.

The reason to call the object defined by this formula a wave function is that it can be shown that at large nucleon separation the interaction between them can be neglected and it behaves like a free wave function. Moreover, it can also be shown [3] that exact scattering phase shifts are encoded in the asymptotics of the NBS wave function. But it contains much more information and motivated by the above wave function interpretation one can define the NBS potential by writing

$$(E_{\mathbf{k}} - H_o)\psi_{\mathbf{k}}^{\mathrm{NBS}}(\mathbf{x}) = U_{\mathbf{k}}^{\mathrm{NBS}}(\mathbf{x})\psi_{\mathbf{k}}^{\mathrm{NBS}}, \tag{2}$$

where

$$E_{\mathbf{k}} = \frac{\mathbf{k}^2}{2M}, \qquad H_o = -\frac{1}{2M}\nabla^2 \tag{3}$$

and $M$ is the reduced mass. This resembles the non-relativistic Schrödinger equation with potential $U_{\mathbf{k}}^{\mathrm{NBS}}(\mathbf{x})$. Indeed, the lattice measurements found that $U_{\mathbf{k}}^{\mathrm{NBS}}(\mathbf{x})$ is very similar to the phenomenological nuclear potential. At large distance it has an attractive tail, but at shorter distances it develops a characteristic repulsive core (RC). While the long distance attraction has long been understood by nuclear theorists and it is due to meson exchanges, it was the first time that the RC has been obtained from a first principles calculation.

Later the same method has been successfully applied also to other hadronic interactions: this included the baryon-baryon potential and the study of 3-body nuclear forces.

Despite these successes, there are also some serious open problems within this approach. The most important difficulty is that, unlike the potential term in the Schrödinger equation, $U_{\mathbf{k}}^{\mathrm{NBS}}(\mathbf{x})$ is energy (momentum) dependent due to the relativistic nature of the problem. Since the energy dependence is weak at low energies, one can define the zero-momentum potential

$$U_o(\mathbf{x}) = \lim_{\mathbf{k}\to 0} U_{\mathbf{k}}^{\mathrm{NBS}}(\mathbf{x}). \tag{4}$$

It can be shown [4] that $U_o$ correctly reproduces the scattering lengths, but already the next to leading order parameter for low energy scattering, the effective range, may differ from the true one.

Motivated by the great importance of the HAL QCD results we decided to investigate two aspects of these researches in some detail. First, to understand the RC better and secondly, to study the energy dependence of the potential. Short distance behaviour of the NBS wave function and potential can be analytically studied, thanks to the asymptotic freedom property of QCD, by operator product expansion and renormalization group techniques [5, 6, 7, 8, 9, 10].

The understanding of the energy dependence is closely related to the problem of trying to find the link between the relativistic field theory description and the non-relativistic concept of potential interaction in the Schrödinger equation. This problem is quite inpenetrable in its original QCD context and for this reason it is better first to model the system in low dimensional toy models. We studied the problem of energy dependence in some $1+1$ dimensional integrable field theory models [4]. In these studies the Ising model and the O(3) nonlinear sigma model were considered and it was found that $U_o(\mathbf{x})$ is indeed a good approximation at low energies where the energy dependence is weak.

A more interesting toy model to study would be the sine-Gordon (SG) model, because unlike in the Ising model and the O(3) model (which are free and repulsive, respectively), here we have both repulsive (soliton-soliton) and attractive (soliton-antisoliton) scattering and in addition there are soliton-antisoliton bound states (breathers). Luckily, an alternative description of the SG model exists since it is known that for any fixed particle number subspace of the SG field theory there is a corresponding Ruijsenaars-Schneider (RS) type relativistic quantum mechanical description [11]. The RS wave function is known [12] for both soliton-soliton and soliton-antisoliton scattering and exactly reproduces the scattering phase shifts of SG field theory. Moreover, also the soliton-antisoliton bound state spectrum is calculable and exactly match the SG results. This lead us to relativistic particle systems, in particular two-particle systems which was the second theme in the research.

We took one more simplifying step and considered the classical relativistic RS 2-particle scattering problem. Energy dependence of the potential is already present in this very simple system but here the problem can be completely solved using textbook results for classical inverse scattering. We can find the relation between the zero-momentum potential and the true effective potential analytically. Later the zero-momentum potential versus effective potential relation can similarly be found in the relativistic RS quantum mechanical problem, using the existing methods of quantum inverse scattering.

The papers relevant for the OTKA grant include [5, 8, 9, 10, 16, 17, 18]. In addition I have given seminar talks on the first problem in Asian countries. In 2013 a seminar at the Physics Department of Tsukuba University, Japan and in 2014 I was invited speaker in the conference "Workshop on hadron physics" in Lanzhou, China.

### Short distance behaviour of the NBS wave function

We can understand the short distance behaviour of the NBS wave functions and the corresponding potentials by combining the OPE (operator product expansion) and a renormalization group analysis in QCD perturbation theory.

The operator product expansion (OPE) method is based on the observation that the product of the two renormalized nucleon operators appearing in the definition of the NBS wave function is for short distances expressed as a sum of local gauge invariant operators times coefficient functions:

$$N_\alpha(0,0)N_\beta(\boldsymbol{x},0) \underset{\boldsymbol{x}\to 0}{\sim} \sum_A F^A_{\alpha\beta}(\boldsymbol{x})\mathcal{O}_A\,. \tag{5}$$

The coefficient functions $F^A_{\alpha\beta}(\boldsymbol{x})$ appearing above can be computed perturbatively using the RG equations which express independence of the bare theory of the renormalization scale.

The leading asymptotic short distance behaviour of the NBS wave function is dominated by the operator with the largest anomalous dimension.

Given an arbitrary basis of gauge invariant $r$-quark operators the anomalous dimensions are related to the eigenvalues of the matrix specifying the mixing under renormalization. For a given set of Dirac and flavor indices the initial set of operators may be quite large. The potential is repulsive at short distances if the largest dimension is negative and attractive if it is positive.

We have considered the following four problems: short-distance interaction between two nucleons [5], between two octet baryons [7], among three nucleons [8] and finally among three octet baryons [9]. In all these cases we mainly concentrated on the question of the existence of a repulsive (or attractive) core of the interaction potential. The main results from these analyses are as follows.

**2-body forces**

The technical part of the computation [5] for these cases consists of two main steps: the enumeration of independent local gauge invariant 6-quark operators, and the calculation of the spectrum of 1-loop anomalous dimensions (by diagonalizing the 1-loop mixing matrix).

*2-nucleon potential (2 flavors) [5]*

The most important property of the spectrum in this case is that the anomalous dimensions of all operators are non-positive. Almost all operators have negative values indicating repulsive behaviour, but there are also operators of zero anomalous dimensions. In these cases whether the leading behaviour corresponds to attraction or repulsion cannot unfortunately be decided using perturbation theory alone. However, we were able to use a simple effective model to conclude [5] that the interaction leads to repulsion in all cases.

*2-baryon potential (3 flavors) [7]*

Again, most operators have negative eigenvalues, but there are also attractive cases, operators with positive values. These all belong to the representations $1_s$, $8_s$ and $8_a$, consistent with what we found for the 2-nucleon case, since 2-nucleon states belong to the $\overline{10}_a$ and $27_s$ SU(3) representations. (Here we indicate the dimension and symmetry property of the relevant SU(3) representation.)

**3-body forces**

Collective phenomena in nuclear physics cannot be described using only 2-body potentials. In particular, the existence of massive neutron stars suggests that also the 3-nucleon potential has a repulsive core. To study 3-body forces with our method we introduced the 3-particle generalization of the NBS wave function.

In the UV limit we can use the OPE method to calculate the leading behaviour of the wave function analogously to the 2-body case but here we have to consider local gauge invariant 9-quark operators. We have carried out the anomalous dimension analysis both in the 3-nucleon case and the most demanding 3-baryon case.

*3-nucleon potential (2 flavors) [8]*

The most striking feature of the spectrum in this case is that for *all* operators, all anomalous dimensions are strictly negative. Thus we conclude that the total local potential corresponds to short-distance repulsion.

We note that showing the presence of a repulsive core in the 3-nucleon case is the most important result of our perturbative considerations. Universal repulsion follows unambiguously without having to rely on any model-dependent results.

We also note that one can observe from the calculated pattern of the spectrum of 1-loop anomalous dimensions that the Pauli exclusion principle is at work here: increasing the number of quarks while keeping the number of flavors fixed makes repulsion stronger while increasing the number of flavors and fixing the number of quarks brings in some attractive channels.

*3-baryon potential (3 flavors) [9]*

The 3-baryon case is the most demanding technically. There are many Dirac index configurations to be considered and for some of them the dimension of the problem is quite large: with Dirac index structure 111223344 for example there are originally 14130 operators. After performing the diagonalization in all Dirac sectors we found that in the 3-baryon case, although the vast majority of the eigenvalues are negative, there are also some attractive channels.

### Relativistic particle systems

With the motivation of understanding the relation between the NBS potential and the phenomenological potential, I was lead to study the problem of formulating relativistic particle interactions without introducing any intermediary fields, as opposed to the usual formulation in the framework of relativistic quantum field theory.

Why this alternative of directly interacting particles was abandoned is due to an apparent conflict with relativistic causality in the case of instantaneous action-at-a-distance interaction among point-like particles. The famous "no-interaction" theorem of Currie, Jordan and Sudarshan states that in a canonical formalism if we represent the 10 generators of the Poincaré group in terms of canonical position and conjugate momentum variables, the Poincaré Lie algebra relations exclude the presence of any non-trivial interaction. This result is intuitively obvious and expresses the apparent incompatibility of causality with particle mechanics.

Soon after formulating the no-go theorem it was realized that the only assumption one has to give up is the canonical behaviour of the position variables and then a consistent fully Poincaré invariant theory describing the trajectories of an isolated system of point-like particles can be established. There are two, essentially equivalent formulations of the same theory. The first formulation is called Predictive Relativistic Mechanics (PRM) and is given by writing equations of motion in a Newtonian form. The accelerations depend on the instantaneous positions and velocities of the particles. Relativistic invariance implies that the accelerations have to satisfy a set of quadratic, partial differential equations, the Currie-Hill (CH) equations [13]. This ensures that if we transform the particle trajectories (obtained by integrating the Newton equations) into a Lorentz-boosted new coordinate system, the particle accelerations in this new system are again satisfying the same instantaneous Newton equations (as function of the positions and velocities in the new system).

Unfortunately no explicit solution of the CH equations are known in the literature. There exist approximate solutions in the $1/c^2$ expansion ($c$ is the speed of light) including the theory of classical electrons either in the Feynman-Wheeler or in Rohrlich's formalism. More important than this academic example are the equations of motion describing compact binaries in General Relativity. These are known in the post-Newtonian (essentially $1/c^2$) expansion up to third order.

Given a solution of the CH equations a natural question is to construct the 10 generators of the Poincaré group and ask if a symplectic structure on the space of trajectories can be found such that the 10 quantities generate the Poincaré group.

An alternative approach to relativistic mechanics [14] can be called canonical. Here a phase space equipped with a symplectic structure is assumed from the beginning, together with the set of 10 generators of the Poincaré group and the Hamiltonian of the model is identified with the generator of time translations from the Poincaré Lie algebra. In this approach the difficulty is to construct the particle positions (trajectory variables) as functions on the phase space. Given the initial positions the dynamics of the system determines the full space-time trajectories and the known action of the Poincaré group on the phase space tells us how the particle trajectories are transformed. In relativistic canonical mechanics we require that this induced action is identical to the usual linear Poincaré transformation of the space-time coordinates corresponding to the trajectories. For infinitesimal transformations the above consistency conditions require that the position variables satisfy the so called the world-line conditions (WLC). If such particle coordinates are found, their Poisson brackets must not vanish, otherwise, due to the no-go theorem, there is no interaction. Again, no explicit solution of the non-linear WLC equations is known. Most constructions [14] are based on constraint dynamics and the trajectory variables are given only implicitly. Nevertheless the canonical approach has the advantage that only the trajectory variables have to be constructed, because the 10 integrals of the Poincaré group are there by construction from the beginning.

Because of the lack of explicit solutions in $3+1$ space-time dimensions it is useful to study toy models in $1+1$ dimensions. Not many solutions are known even for the $1+1$ dimensional analog of the CH equations. Although the most general 2-particle solution has been found in $1+1$ dimensions [15], but it is given in a very implicit form. In [16] a completely explicit solution of the Currie-Hill equations in $1+1$ dimensional Minkowski space-time was presented. This solution can be written in terms of elementary functions and provides an example in which important questions of the relativistic action-at-a-distance approach (conserved quantities, canonical structure, etc.) can be studied transparently.

The most famous $1+1$ dimensional examples are the exactly solvable Ruijsenaars-Schneider (RS) models [11], the relativistic generalizations of the Calogero-Moser systems. The RS approach is canonical, and these systems are important, because they are not only relativistic, but also integrable.

We presented [17] a general algorithm to construct trajectory variables satisfying the $1+1$ dimensional world-line conditions in canonical relativistic models. The algorithm is a generalization of that used in RS models but it is also useful for the RS models themselves. It provides a simple proof of the fact that the WLC are satisfied and this demonstrates true Poincaré invariance for this family of models. Moreover, the non-vanishing of the coordinate Poisson brackets is also shown.

### Effective potential [18]

COM dynamics of the 2-particle Ruijenaars-Schneider model is equivalent to a fictitious non-relativistic problem, except for rescaling with a state-dependent constant of motion. It can be shown that the physical (relativistic) scattering data are simply related to the ones calculated in this NR problem.

Since the relativistic and NR scattering data are very similar, the following question arises naturally. Is there a NR effective potential $W^{\text{eff}}(x)$ such that the scattering phase shifts (and in case of bound motions, the periods) are exactly reproduced by using a non-relativistic Hamiltonian with

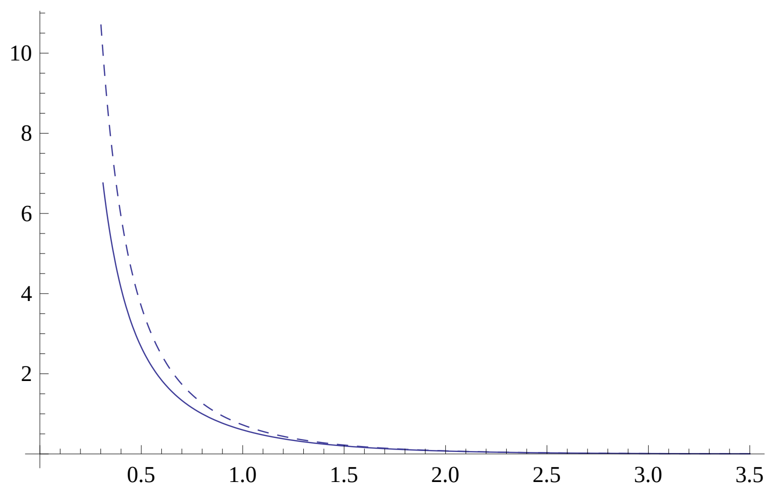


Figure 1: Sine-Gordon effective potential (solid line). The dashed line is the corresponding zero-momentum potential. The plots show $W^{\text{eff}}$ vs. $x$ in natural units.

potential $W^{\text{eff}}$?

For the SG soliton-soliton scattering, the answer is yes. We can use classical inverse scattering to obtain the effective potential, which is given by an integral formula. The integral cannot be calculated analytically, but it is easily obtained by numerical integration. The result is shown in Fig. 1. The leading large distance behaviour is the same as for the zero-momentum potential, but the subleading terms differ.

For the Sine-Gordon soliton-antisoliton problem the answer is no. It can be shown that for attractive potentials there is a constraint between the scattering and bound state data and in this case the constraint is not satisfied. Therefore no $W^{\text{eff}}(x)$ exists.

For potentials with a RC the answer is again yes. We have to use both scattering and bound state data to reconstruct $W^{\text{eff}}(x)$.

It is likely that quantum inverse scattering can be applied to study the same questions at the quantum mechanical level in SG/RS theory. Whether the zero-momentum potential $\rightarrow$ effective potential mapping exists in the physically relevant $3+1$ dimensional nucleon problem is an open question.

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