REPORT

ON THE PROJECT NO. SAB81142

A NEW APPROACH TO ALMOST FACTORIZABILITY

In the structure theory of inverse semigroups, there are two approaches, basically from the 1970's, to build up inverse semigroups from semilattices and groups via their semidirect products. These approaches are dual to each other in the sense that one produces any inverse semigroup from a semidirect product of a semilattice by a group by taking an (idempotent separating) homomorphic image of an inverse subsemigroup, and the other one by taking an inverse subsemigroup of an (idempotent separating) homomorphic image of such a semidirect product, see [20]. The intermediate classes are just the E-unitary inverse semigroups and the almost factorizable inverse semigroups, respectively. The two approaches can be visualized as follows:

$$\begin{tikzcd}
\text{E-unitary inv.sgp.} \arrow[r, hook] \arrow[d, "\mathit{id.sep.}"'] & \text{s.dir.pr. of sl. by gp.} \arrow[d, "\mathit{id.sep.}"] \\
\text{inv.sgp.} \arrow[r, hook] & \text{alm.fact. inv.sgp.}
\end{tikzcd}$$

Note that, in the 1970's, the above picture was established only for inverse monoids, with factorizable inverse monoids in the place of almost factorizable ones. The notion of almost factorizable inverse semigroups was introduced only about twenty years later in order to complete the above picture in the semigroup case — although certain main ideas and partial results were implicit much formerly. While factorizability is easily defined by a simple property of the elements, the definition of almost factorizability is more involved since it makes use of either the monoid of permissible subset or the monoid of certain translations.

An interesting additional structural result for inverse semigroups is that, given an inverse semigroup S, each E-unitary inverse semigroup which has S as an (idempotent separating) homomorphic image — i.e., each E-unitary cover of S — can be constructed, up to isomorphism, from an embedding of S into an almost factorizable inverse semigroup in a natural way.

Since the 1980's, a number of structure theorems have been obtained generalizing the first approach. The generalizations go in two main directions. In one of them the regularity of the semigroups considered is retained and the condition that the idempotents commute is weakened. In the other one commutation of idempotents is retained but regularity is weakened. More precisely, orthodox and locally inverse semigroups are investigated on the one hand, see [3], [4], [22], [27], [28], [29], [33], and (left) ample and weakly (left) ample semigroups on the other hand, see [2], [5], [10], [11], [13], [14], [19]. The main question in these investigations is to determine which semigroups are embeddable into a semidirect or a semidirect-like product of a special kind.

The second approach seems to be more difficult to generalize, and till now this approach is extended only for more restricted classes. Within the class of orthodox semigroups, almost factorizability is defined by Hartmann [16] based on translations, and it is shown that an orthodox semigroup is almost factorizable if and only if it is an idempotent separating homomorphic image of a semidirect product of a band by a group. It is also valid ([17]) that each orthodox semigroup is embeddable into an almost factorizable one. A notion of almost factorizability is introduced for straight locally inverse semigroups by Dombi [7], based on an appropriate notion of permissible sets introduced in [24], and important properties — reminiscent to the inverse case — of almost factorizable straight locally inverse semigroups are obtained. Recently, we have found in [30] a concept of almost factorizability within the class of all locally inverse semigroups by giving up that the definition be based, in any sense, on permissible sets or translations, but in a way that the most important results known formerly in the special cases remain valid in the general case.

A notion of, necessarily one-sided, factorizability is introduced within the class of left ample monoids by El Qallali [8], who together with Fountain [9] studies the covers obtained from factorizable embeddings. Gomes and the principal investigator [15] study the same notion of left factorizability within the class of weakly ample monoids, and introduce a corresponding notion of almost left factorizability within the class of weakly ample semigroups, based on a concept of permissible sets taylored for this class. Surprisingly, the class obtained turns out to coincide with a class stemming from former results of the first approach on left ample semigroups, and the structure of the members shows analogy with the inverse case. However, we have to emphasize that these notions of factorizability have not been motivated by the idea of extending the second approach for these classes.

The aim of the present project was to generalize the second approach for classes of non-regular generalizations of inverse semigroups where the first approach is (at least partly) developed; e.g. for (left) ample or, possibly, for weakly (left) ample semigroups. Note that the intersection of the classes of left ample and of locally inverse semigroups is just the class of inverse semigroups, that is, these classes generalize the inverse semigroups in two 'opposite' directions. Thus the ideas and methods required in the research area of the project are completely different from those used for locally inverse semigroups.

The results achieved in the project are valid for the classes of restriction and left restriction semigroups which form proper superclasses of weakly ample and weakly left ample semigroups, respectively. The notion of a (left) restriction semigroup generalizes that of a weakly (left) ample semigroup in a way that the role played by the semilattice of idempotents in a weakly (left) ample semigroup is taken over in a (left) restriction semigroup by a semilattice of distinguished idempotents called projections. However, (left) restriction and weakly (left) ample semigroups are close to each other in the sense that, in most cases, the arguments applied in the weakly (left) ample case go through also in the (left) restriction case. Therefore we allow us in the sequel to formulate results for (left) restriction semigroups although they are proved in the literature only for weakly (left) ample semigroups.

Left ample semigroups and, more generally, weakly left ample semigroups have been attracted considerable attention in the structure theory of semigroups since their appearence in [10]. However, they were investigated — under different names, and with motivation coming from various topics — much earlier and also later on in other areas of semigroup theory, see [1], [18], [21], [25], [26], [34]. Recently, they turned up also in theoretical computer science ([6]).

The results for inverse semigroups we summarized above as the first approach is generalized for left ample semigroups in two different ways: the semidirect product of a semilattice by a group is replaced, on the one hand, by the semidirect product of a semilattice by a right cancellative monoid ([2]), and on the other hand, by a certain subsemigroup of the reverse semidirect product of a semilattice by a right cancellative monoid ([11]). For brevity, we refer to the latter construction as a W-product of a semilattice by a monoid. These two ways are further generalized for weakly left ample semigroups in [14] and [5], respectively. It was the W-product construction of a semilattice by a monoid that played a crucial role in the main result of [15] mentioned above and formulated as follows: the almost left factorizable weakly ample semigroups are just the (idempotent separating) homomorphic images of the W-products of semilattices by unipotent monoids.

Thus the results known before the present project in the class of left restriction semigroups and in that of restriction semigroups can be visualized as follows:

$$\begin{tikzcd}
\begin{array}{r}\text{proper}\\ \text{left.restr.sgp.}\end{array} \bullet \arrow[r, hook] \arrow[d, two heads, "pr.sep."'] & \bullet \begin{array}{l}W\text{-pr. of}\\ \text{sl. by mon.}\end{array} \\
\text{left.restr.sgp.} \bullet &
\end{tikzcd}$$

$$\begin{tikzcd}
\begin{array}{r}\text{proper}\\ \text{restr.sgp.}\end{array} \bullet \arrow[d, two heads, "pr.sep."'] & \bullet \begin{array}{l}W\text{-pr. of}\\ \text{sl. by sgp.}\end{array} \arrow[d, two heads, "pr.sep."] \\
\text{restr.sgp.} \bullet & \bullet \begin{array}{l}\text{alm.l.fact.}\\ \text{restr.sgp.}\end{array}
\end{tikzcd}$$

Note that 'properness' is the property taking over the role of 'E-unitariness' within these non-regular generalizations of inverse semigroups.

The main results achieved in the project concern restriction semigroups:

Theorem 1. *For each restriction semigroup S, there exists a proper restriction semigroup — what is more, a proper ample semigroup — T such that S is a projection separating homomorphic image of T, and T is embeddable into a W-product of a semilattice by a monoid.*

Theorem 2. *Each restriction semigroup is embeddable into an almost left factorizable restriction semigroup.*

This completes the above picture for restriction semigroups:

$$\begin{tikzcd} \text{proper restr.sgp.} \arrow[r, hook, dotted] \arrow[d, "\mathit{pr.sep.}"'] & W\text{-pr. of sl. by sgp.} \arrow[d, "\mathit{pr.sep.}"] \\ \text{restr.sgp.} \arrow[r, hook, dotted] & \text{alm.l.fact. restr.sgp.} \end{tikzcd}$$

In [13], free restriction semigroups are described, and it is noticed that each restriction semigroup is a projection separating homomorphic image of a so-called quasi-free restriction semigroup. A quasi-free restriction semigroup is a factor semigroup of a free one over a special congruence that we call in this report naturally ample. This name refers to the property that quasi-free restriction semigroups are ample, and that these congruences are 'natural' among those producing an ample factor. The proof of Theorem 1 goes in the following way. First, starting from that in [13], we give an alternative model $F_W\mathcal{RS}(X)$ for the free restriction semigroup on X which is a subsemigroup in a W-product $W(X^*, \mathcal{Q})$. Here $\mathcal{Q}$ is a semilattice whose elements are certain finite connected subgraphs in the Cayley graph of the free group on X, and X^* is the free monoid on X. Second, we provide a combinatorial description of any natural ample congruence ρ of $F_W\mathcal{RS}(X)$ by means of a congruence ν_ρ of $\mathcal{Q}$. This description is transparent and simple enough to work with. In particular, it allows us to prove that the quasi-free restriction semigroup $F_W\mathcal{RS}(X)/\rho$ is embeddable into the W-product $W(X^*, \mathcal{Q}/\nu_\rho)$. So an important byproduct of the proof is the following result.

Proposition 3. *Each quasi-free — in particular, each free — restriction semigroup is embeddable into a W-product of a semilattice by a free monoid.*

The latter result forms a starting point to Theorem 2, since it provides embeddability into almost left factorizable restriction semigroups for a subclass of restriction semigroups. (However, notice that Theorem 2 does not imply Proposition 3.)

Any restriction semigroup is isomorphic to a factor semigroup $F_W\mathcal{RS}(X)/\tau$ for some set X and congruence τ. Since $F_W\mathcal{RS}(X)$ is a subsemigroup in $W(X^*, \mathcal{Q})$, the relation τ generates a congruence $\tau^\#$ on $W(X^*, \mathcal{Q})$, and there is a natural homomorphism from $F_W\mathcal{RS}(X)/\tau$ into $W(X^*, \mathcal{Q})/\tau^\#$. In particular, if the restriction of $\tau^\#$ to $F_W\mathcal{RS}(X)$ coincides with τ then this homomorphism is an embedding. Since $W(X^*, \mathcal{Q})/\tau^\#$ is an almost left factorizable restriction semigroup, in order to prove Theorem 2, it suffices to show the following statement having some interest in itself.

Proposition 4. *Let X be a non-empty set, and let τ be a congruence on $F_W\mathcal{RS}(X)$. Consider the congruence $\tau^\#$ on $W(X^*, \mathcal{Q})$ generated by τ. Then the restriction of $\tau^\#$ to $F_W\mathcal{RS}(X)$ is equal to τ.*

This is the initial idea of the proof of Theorem 2. We have to emphasize that restriction semigroups are algebras with a binary and two unary operations, therefore subsemigroups, homomorphisms (in particular, embeddings), congruences and factor semigroups are understood in this signature. In particular, the congruence $\tau^\#$ on $W(X^*, \mathcal{Q})$ can be obtained from τ by making use of unary polynomials in this signature. Most of the proof of Proposition 4 is devoted to reducing the set of unary polynomials needed. Reduction consists of two steps where the first one is valid for any restriction semigroup. The second one concerns only $W(X^*, \mathcal{Q})$, and it is taylored to the goal of proving Proposition 4. Again, the arguments in the latter part are combinatorial, and the graphs belonging to $\mathcal{Q}$ play a crucial role.

These results are submitted for publication: Theorem 1 (together with Proposition 3) is written up in [31], and Theorem 2 (together with Proposition 4) in [32].

After having the complete rectangular diagram for restriction semigroups at hand, the next question motivated by the inverse case concerns the connection between proper covers and embeddings into almost left factorizable restriction semigroups. We have started to study in a joint research with G. Gomes under what conditions a proper cover of a restriction semigroup S can be constructed, up to isomorphism, from an embedding of S into an almost left factorizable restriction semigroup. A necessary condition is that the cover be embeddable in a W-product of a semilattice by a monoid. However, we have not found a necessary and sufficient condition yet.

In the one-sided case, that is, in the class of left restriction semigroups we have also achieved partial results in the framework of the project. Namely, as a preparation for being able to characterize the homomorphic images of W-products of semilattices by monoids, we constructed the 'free' ones among all such W-products. Notice that homomorphic images are meant here in the signature of left restriction semigroups, that is, of algebras with a binary and only one unary operation. So the class we are looking for here is much wider than that of all almost left factorizable restriction semigroups (considered as left restriction semigroups). Another question we have studied with V. Gould askes under what conditions a proper left restriction semigroup is embeddable in a W-product of a semilattice by a monoid. Here we have already found necessary and sufficient conditions. At present we are working on simplifying these conditions.

During the running time of the project, the principal investigator gave four scientific talks. Two of them were in the topic of the project:

- Some results on almost factorizable semigroups (24 Nov, 2010, Meeting NBSAN 2010, York),
- On a generalization of inverse semigroups (23 May, 2011, Algebra Seminar, Rényi Institute, Budapest);

and the rest in other topics:

- Factorizability for locally inverse semigroups (25 Oct, 2010, Algebra Seminar, Rényi Institute, Budapest),
- Finite F-inverse covers of finite inverse monoids (30 Nov, 2010, Seminar on Semigroups, University of York, York).

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