According to our original proposal our research is focused on the following areas:

- General theory of composite beams,
- Local buckling analysis of thin walled beams,
- Vibration of composite floors supported by beams.

Later we included in our research the

• investigation of RC columns strengthened with composite materials and the vibration of arches and columns;

and - with permition - Bernat Csuka and Tamas Ther joined our team.

In all four areas we fulfilled our goals, as it will be discussed below. It must be admitted, however that the whole research budget was not spent, the reason is that some of the conference participations were financed by other sources, and the PI – because of other commitments – had much less time for international travelling.

The most important achievement of our research is the new beam theory (first topic), this will be discussed in detail, then the other topics are presented briefly. The details can be found in the referred articles.

## GENERAL THEORY OF COMPOSITE BEAMS.

When composite beams and columns are designed the stresses and strains are calculated, and the buckling loads and the natural frequencies are determined. In the analysis the cross sectional properties (the beam's stiffness matrix) must be known. Its calculation can be more complex for composite beams than for isotropic ones, because of the substantial differences in their response to external loads. The behaviour of isotropic and composite beams is illustrated below with the example of a thin walled I-beam.

When an *isotropic* beam is subjected to tension or bending the cross sections remain plane (Figures 1a and b), while under torsion cross sections warp (Figure 1c).



Figure 1. Deformation of an isotropic I-beam subjected to (a) tension, (b) bending and (c) torque

In the presence of structural constrains the problem of torsion is more complex and – for a built in I-beam – can be illustrated as the loading of the flanges by a force couple (Figure 2a). Due to these loads the flanges undergo bending deformations (when the shear deformations of the flanges are neglected, Figure 2a) and shear deformations (Figure 2b). Note that for long beams the shear deformations of the flanges are negligible. For isotropic beams the tension, bending and torsion are uncoupled.



**Figure 2.** Deformations of a built-in isotropic I-beam subjected to torsion when (a) the shear deformations of the flanges are neglected and (b) due to the shear deformations of the flanges



**Figure 3.** Possible deformations of a composite I-beam subjected to tension. (a) the warping of the cross section, (b) the tension-torsion coupling and (c) the shear deformation of the



**Figure 4.** a) Example of an I-beam with unbalanced flanges. (b) Under pure tension there is no twist, however, the cross sections warp. (c) When one end is fixed, the beam subjected to tension will twist.

For thin-walled *composite beams* the following phenomena may occur which are significantly different from the behaviour of isotropic beams: the cross section may warp under pure tension or pure bending (the first one is illustrated in Figure 3a), there are tension-torsion, tension-bending and bending-torsion coupling (the first one is illustrated in Figure 3b), there is a tension-shear coupling in the flanges as illustrated in Figure 3c.

We further illustrate the difference between isotropic and composite beams when both tensionshear coupling (Figure 3c) and structural constrains are present. We consider the example of an I-beam with unbalanced flanges (e.g. the fibers in the upper flange are in the  $+45^{\circ}$ , while the fibers in the lower flange are in the  $-45^{\circ}$  direction, Figure 4a). Under pure tension the flanges undergo shear deformation (the cross section warps), however, there is no twist along the beam (Figure 4b) and there is no tension-twist coupling. When one end is built-in and the other end is free, under tension the rotation of the beam will be significant due to constrained warping (Figure 4c). Note that this effect is significant also for long beams.

*Isotropic* beam theories and the corresponding computer codes can take into account all the effects explained in Figures 1 and 2 except the shear deformation due to torsion and structural constraint (Figure 2b), note, however, that for isotopic beams this effect is negligible.

*Anisotropic* beam theories (and the corresponding computer codes) can handle the warping of the cross sections (Figure 3a), which means that cross sections do not remain plane under pure tension and bending, and also the coupling among tension, torsion and bending, however they do not include

- the shear deformation of the flanges due to restrained warping (Figure 2b),
- the tension-shear coupling in the flanges (Figure 3c).

We emphasize that for composite beams the restrained warping induced shear deformations are important and – except for long beams – can not be neglected [1]. In addition, when tension-shear coupling is present (Figure 4b and c) theories which do not include this effect may lead to unacceptable results even for long beams.

In this research we present an analysis of thin-walled composite beams, taking into account restrained warping induced shear deformation and the tension-shear coupling. We will summarize only the basic idea of the method to calculate the *stiffness matrix* of composite beams, the details of the analysis is presented in [2].

Beam theories give the relationships between the *displacements* of the beam axis, generalized *strains*, internal *forces* and *loads*. These relationships are given by the strain-displacement relationships (geometrical equations), material law (constitutive equations) and the equilibrium equations. These are written as:



Figure 5. The stress resultants (internal forces).



Figure 6. Illustration of Saint Venant torque, restrained warping induced torque and bimoment on an I-beam.

where  $\mathbf{u}, \mathbf{\epsilon}, \mathbf{N}, \mathbf{p}$  are the vectors of the displacements, generalized strains, internal forces and

loads, respectively. **M** is the (symmetric) stiffness matrix, while  $\hat{\Theta}$  and  $\hat{\Theta}^*$  are operator matrices. For the *spatial* case the minimum number of internal forces is six due to the six stress resultants (Figure 5): the axial force, the two transverse shear forces, the two bending moments and the torque. (By neglecting the shear deformations the shear forces can be eliminated, but for composites – as it is discussed in [1] – this is not recommended.) It is well known that for open section beams the theory including these six forces only is inaccurate, and the torque must be divided as Saint Venant and restrained warping induced torque:  $T_{SV} + T_{\omega}$  (Figure 6), where the latter one is the derivative of the bimoment (or moment couple),  $T_{\omega} = \partial M_{\omega} / \partial x$ . The constitutive equations of this theory – often referred to as Vlasov's theory – are given in the top part of Table 1. In this theory six displacements must be taken into account: u, v and w are the displacements of the axis,  $\psi$  is the rotation of the cross section about the beam's axis and  $\chi_v$ 

,  $\chi_z$  are the rotation of the cross sections about the z and y axes. For cross sections made of isotropic or orthotropic materials only some of the elements in the stiffness matrix are zero [1], which are denoted by stars in Table 1.

#### Classical theory



Torsional-warping shear deformation theory



**Table 1.** Constitutive equations for the spatial problem (Stars in the  $M_{or}$  matrix show the nonzero elements in the stiffness matrix of an orthotropic beam.)

*Shortcomings of the classical theory.* It is well known that composite structures undergo higher shear deformation than structures made of conventional materials, and hence the shear deformation should not be neglected. For restrained warping the torsional shear deformations must also be taken into account. This is illustrated in Figure 2 for an orthotropic I beam. According to Vlasov's theory, when an I beam is subjected to torsion there is no shear deformation of the flanges (Figure 2a), when, in fact, the shear deformation (illustrated in Figure 2b) may be significant. This effect can be modelled by the *torsional-warping shear deformation theory*, which is developed in this OTKA project. It can be argued that for longer beams this effect is negligible. This may be true for orthotropic beams, but not for anisotropic ones, as it was shown in Figure 4.

#### Torsional-warping shear deformation theory.

To understand the torsional-warping shear deformations, first we consider the case when a *beam deforms only in a plane* (e.g. in the *x*-*y* plane). In the classical beam theory [3], (when the shear deformation is neglected), the displacements of the axis in the *x*- and *y* directions (*u* and *v*) are used to calculate the strains and deformations of any point of the cross section. When the shear deformation is taken into account, according to Timoshenko's beam theory (see Kollár and Springer [1] for composite beams), three displacement functions of the axis are required: the displacement along and perpendicular to the axis (*u* and *v*) and the rotation of the cross section  $(\chi_v)$ .

When a *beam is subjected to torsion*, in the classical (Vlasov or Wagner) theory only the rotation of the cross section  $(\psi)$  about the beam's axis is used [3], [4] to calculate the displacements of any point of the cross section. When the axial warping is constrained, an open

section beam carries the torque load mainly by the bending and shear of the flanges, as illustrated in Figure 2 for a symmetrical I-beam. Note, however, that according to Vlasov's theory the shear deformations of the walls (Figure 2b) are neglected. To overcome this shortcoming, analogously to Timoshenko's beam theory, we introduced [5] a new displacement function (in addition to the rotation of the cross section,  $\psi$ ): the rate of twist due to warping (

 $\vartheta^{B}$ ). In other words, the rate of twist  $(\partial \psi / \partial x)$  consists of two parts, one when the shear deformation is zero and one when the warping is zero:

$$\frac{\partial \psi}{\partial x} = \vartheta^{\mathrm{B}} + \vartheta^{\mathrm{S}} \,. \tag{2}$$

We must give credit to Wu and Sun [6], who suggested first the introduction of this new function.

In summary, in this theory seven displacements must be taken into account: u, v, w,  $\psi$ ,  $\chi_y$ ,  $\chi_z$ ,  $\vartheta^{\rm B}$ , where u, v and w are the displacements of the axis,  $\psi$  is the rotation of the cross section about the beam's axis,  $\chi_y$ ,  $\chi_z$  are the rotation of the cross sections about the z and y axes, and  $\vartheta^{\rm B}$  is the rate of twist due to warping. This theory was developed for *orthotropic* open [5] and closed [7] section beams, the stiffeness matrix is given in Figure 7.



Figure 7. Stiffness matrix of Vlasov-, Timoshenko- and torsional-warping shear deformation theory.

The new element in the stiffness matrix is the rotational shear stiffness  $S_{\omega}$ . For a few cases analytical expressions are given for the calculation of the stiffnesses of orthotropic composite thin walled beams including the rotational shear stiffness [5],[7].

For anisotropic beams there are  $8 \times 8$  elements of the (symmetric) stiffness matrix if the torsional-warping shear deformation theory is used. There are several methods, which can be used to determine the stiffnesses of a beam [2], [8-14].

#### Method of Solution

The basic idea of calculating the stiffenesses is as follows. The displacements are assumed in the form of sine (or cosine) functions:  $\sin \alpha x$ ,  $\cos \alpha x$  ( $\alpha = \pi/L$ ). Then the strains, internal forces and loads are determined. Each contains either sine or cosine functions only. Now the average strain energy per unit length of the structure is determined for the beam model and also for the 3D model. Both are function of 1/L. The stiffness matrix **M** is determined from the condition that the coefficients of the Taylor series expansions of the strain energies are equal. The key of the solution is that for trigonometrical functions the differential equation system can

be replaced by ordinary (matrix) equations: If  $\sin \alpha x$ ,  $\cos \alpha x$  ( $\alpha = \pi/L$ ) is differentiated with respect to x the result is a trigonometrical function multiplied by  $\alpha$ . As a consequence, for trigonometrical displacements the differential equation system Equation (1) can be replaced by ordinary equations, where the coefficient matrices contain  $\alpha$ . The details of the solution are given in [2].

We developed a computer code, designated as BEAMSIN to calculate the stiffnesses of thin walled anisotropic beams. The code is based on the 3D analytical solution of beams presented in [2]. When the stiffnesses are known the displacement can be calculated either numerically (FE) or analytically [2]. Here only the results of one example [2] are presented for an I-beam cantilever with unbalanced flanges subjected to a tensile load (Figure 4c) which causes the rotation of the cross section. These calculations were compared to the results of a finite element (ANSYS) analysis, where shell elements were used. The solutions of the VABS analysis [13], [14] are also included. VABS does not contain the restrained warping induced shear deformations, hence it cannot predict well the tension – warping-shear coupling.



**Figure 8.** Rotation  $\psi$  of an I-beam cantilever with unbalanced flanges subjected to tensile force (Figure 4c)

#### LOCAL BUCKLING ANALYSIS OF THIN WALLED BEAMS

Local buckling analysis is a major consideration in the design of thin-walled FRP sections. Local buckling analyses of members can be performed by modeling the wall segments as orthotropic plates and by assuming that edges common to two or more plates remain straight. Then the buckling load is determined either (i) "exactly" (assuming that all the wall segments buckle simultaneously and the continuity conditions at the plate intersections are satisfied) or (ii) approximately, by considering the wall segments as individual plates, which are elastically restrained by the adjacent walls (Figure 8) [15-21].

When the second method is used (e.g., only one of the wall segments is considered, which is elastically restrained by the adjacent walls) the two keys to the solution are (a) to determine the elastic restraints caused by the adjacent walls (see [1, 21]), and (b) to calculate the buckling load of the plate, whose edges are restrained.



Figure 9. Web with restrained edges. The restraining wall segment may have (a) two edges attached to adjacent walls or (b) one edge is free

The buckling analyses of long orthotropic plates with different edge conditions were treated by several authors; however closed form expressions were available only for a few cases ([1]).

Explicit expressions were published for the calculation of the buckling loads of long plates subjected to *bending or shear* (Figure 10), when the long edges are elastically restrained [23]. These cases are important for calculating the web buckling of beams subjected to bending or transverse loads.



**Figure 10.** The considered plate (a) and its loads: uniform compression (b), linearly varying compression (i.e. bending) (c), shear (d)

As a consequence, in the analysis of local buckling of FRP members [15] axial loads could be treated with sufficient accuracy, however the web buckling of transversely loaded beams could be approximated only by neglecting the restraining effects of the flanges [15].

In this research analysis of the web buckling of transversely loaded beams is developed. The results are presented in [22-25].

Using these expressions the local buckling analysis of thin walled composite beams given in the literature can be extended to all practical cases. In the derivation of the presented expressions several approximations were made:

- (1) Rather than considering an assembled section, we modeled the webs and flanges as individual orthotropic plates rotationally restrained at their edges.
- (2) In calculating the restraining effect we assumed cylindrical bending.
- (3) The stabilizing effect of tension was neglected.
- (4) We calculated the buckling loads of plates with rotationally restrained edges by approximate expressions.
- (5) The distribution of the shear along the web is assumed to be uniform, the shear in the flanges is neglected
- (6) The maximum values of the internal forces are taken into account, however, in most practical cases they vary along the length

Approximations (2), (3) and (6) are on the safe side, while approximations (4) may cause an error less than 7%. Note that approximation (6) may underestimate the buckling load by up to 10-20%.

Approximation (2) can be eliminated by taking into account the effect of the buckling length in the calculation of the spring constant, however it requires a rather complex calculation, and hence it is not recommended.

We emphasize that several comparisons were made with the results of finite element and analytical calculations, and we found the presented expressions to be reasonably accurate.

### VIBRATION OF COMPOSITE FLOORS SUPPORTED BY BEAMS

Long-span (reinforced concrete, steel, composite, timber- and timber-concrete) floors may show considerable vibration, which may disturb the occupants. The floor is often supported by columns and beams, which may reduce the natural frequency even in the range of human excitation. In this research a model and simple explicit expressions are developed for the calculation of the natural frequency of plates, which take into account the deflections of the supporting beams. The floor is modelled as an orthotropic plate, while the effect of the supporting beams is taken into account either with Föppl's expression or by the Rayleigh-Ritz method. The results are applied to timber and timber-concrete floors with various configurations, and the results are verified numerically and experimentally. The results are summarizes in two mayor journal articles [26], [27].

# INVESTIGATION OF RC COLUMNS STRENGTHENED WITH COMPOSITE MATERIALS AND THE VIBRATION OF ARCHES AND COLUMNS

A new model for FRP confined circular concrete columns based on a sophisticated material model is derived. With the aid of this model the effect of the stiffness of the confining material on the strength of the structure was investigated. It was found that: (i) in the case of a wide parameter range (low stiffness confinement) the stiffness has a minor effect on the concrete strength, (ii) in the case of high stiffness confinement a significant gain in concrete strength can be reached by taking into account the confinement stiffness, and (iii) in theory the concrete can be overconfined (with a lower strength), however this case is not realistic for conventional FRP. Based on the new model an analytical expression is derived to determine the (lower limit of) strength of confined concrete, and the limit of insufficient conf inement is also derived. The results are verified by experiments available in the literature. The results are summarized in four mayor journal articles [28-31], and in the PhD thesis of Bernat Csuka.

In Hungary, several churches were built in the XII – XIX centuries with vaults supported by arches, many of them were severely damaged by moderate ground motions. It is essential that these structures should be evaluated for the expected seismic event. It is well known, that the classical analysis used for the design of regular building by the engineers, such as the Response Modal Analysis or even the time history analysis of elasto-plastic structures are not applicable for masonries, where the "rocking" (opening and closing with impact) plays an important role in the nonlinear response of masonry structures. This is the reason why several researchers were investigated this problem either with the discrete element method or with simplified analytical solutions taking into account the impact between the elements. In this research, first experiments were performed at the laboratory of the Budapest University of Technology and Economics for columns made of dry-masonry. The motions have been recorded with a camera, and the motions of the bricks have been determined by image-processing. Our preliminary results were published in a conference paper [33]. We ate still working in this area, this is the PhD topic of Tamas Ther, who will defend his thesis next year.

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