RESEARCH SUMMARY OF THE OTKA PROJECT PD 77604 "HIGHER-DIMENSIONAL COMPLEX ALGEBRAIC GEOMETRY"

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This is the summary of the research project PD 77604 "Higher-dimensional complex algebraic geometry" supported by the OTKA that took place between April 2009 and February 2011. According to the original plan, the duration of the project would have been 36 months, which in turn got shortened to 23 months due to an extended leave of the principal investigator. This should be taken into account during the evaluation process. Also, as a result, there will be additional publications appearing in the future that will be financed in part by the project.

The list of publications that appeared in the above-mentioned period goes as follows.

(1) Alex Küronya, Victor Lozovanu, Catriona Maclean: *Examples of Okounkov bodies*, in preparation, available from the homepage `http://www.math.bme.hu/~kalex`.
(2) Th. Bauer, C. Bocci, S. Cooper, S. Di Rocco, B. Harbourne, K. Jabbusch, A. Küronya, A. L. Knutsen, R. Miranda, H. Schenk, T. Szemberg, Z. Teitler: *Recent developments and open problems in the theory of linear series*, preprint, `arXiv:1101.4363`.
(3) Küronya Alex: *Bevezetés az algebrai kombinatorikába*, electronic lecture notes, 2011, 167 pp.
(4) Alex Küronya: *Positivity on subvarieties and vanishing theorems for higher cohomology*, preprint, `arXiv:1012.1102`.
(5) Alex Küronya: *Arithmetic properties of volumes of divisors*, Oberwolfach Report No. 45/2010
(6) Alex Küronya, Victor Lozovanu, Catriona Maclean: *Convex bodies arising as Okounkov bodies of divisors*, preprint, `arXiv:1008.4431`.
(7) Alex Küronya, Victor Lozovanu, Catriona Maclean: *Volume functions of linear series*, preprint, `1008.3986`.
(8) Alex Küronya: *Okounkov bodies in low dimensions*, Oberwolfach Report No. 21/2009, 1139–1142

Introduction

The purpose of the project was to study various phenomena in higher-dimensional complex geometry associated to linear series on varieties. The significance of positive line bundles in geometry is widely accepted by now. The original concept of positivity for line bundles is ampleness, which in essence means that one can use the global sections of some tensor power of the given bundle to construct an embedding of the underlying variety into some projective space. Over the years various equivalent characterizations have been found, from which an extremely useful tool arose with cohomological, geometric, and numerical descriptions. The fundamental result in this direction is the theorem of Cartan–Serre–Grothendieck (see [24, Theorem 1.2.6]): let X be a complete projective scheme, $\mathcal{L}$ a line bundle on X. Then the following are equivalent:

(1) There exists a positive integer $m_0 = m_0(X, \mathcal{L})$, such that $\mathcal{L}^{\otimes m}$ is very ample for all $m \geq m_0$ (that is, the global sections of $H^0\left(X, \mathcal{L}^{\otimes} m\right)$ give rise to a closed embedding of X into some projective space).
(2) For every coherent sheaf $\mathcal{F}$ on X, there exists a positive integer $m_1(X, \mathcal{F}, \mathcal{L})$ for which $\mathcal{F} \otimes \mathcal{L}^{\otimes m}$ is globally generated for all $m \geq m_1$.
(3) For every coherent sheaf $\mathcal{F}$ on X there exists a positive integer $m_2(X, \mathcal{F}, \mathcal{L})$ such that

$$H^i\left(X, \mathcal{F} \otimes \mathcal{L}^{\otimes m}\right) = 0$$

for all $i \geq 1$ and all $m \geq m_2$.

Extremely important as they are, ample divisors have various shortcomings. For one, many effective divisors found in nature are not ample, which becomes increasingly important when it comes to birational geometry. Namely, the pullback of an ample line bundle under a proper birational morphism (which leaves the underlying variety unchanged away from a codimension two locus), is no longer ample.

For these and many other reasons it makes sense to study line bundles, which, although not ample themselves, still retain some amount of positivity. There are various ways to make this requirement precise, the most common is to consider line bundles whose tensor powers give rise to birational maps instead of closed embeddings. Such bundles are called big; a famous observation of Kodaira shows that a divisor D is big if and only if some multiple mD is linearly equivalent to the sum of an ample and an effective divisor:

$$mD \sim A + E \ .$$

The behaviour and invariants of big line bundles have been a subject of intensive study in the last decades, for overviews, see [8] or [24, Chapter 2], and the references therein.

After many years of intensive study it comes somewhat as a surprise that some fundamental questions about big line bundles are not settled. In fact, the following direct generalization of the Cartan–Serre–Grothendieck characterization (2) of ample line bundles proven in [22] has not been previously known.

Theorem 0.1. *(Küronya, [22]) Let X be an irreducible projective variety, L a Cartier divisor on X. Then L is big precisely if there exists a proper Zariski-closed subset $V \subsetneq X$ such that for all coherent sheaves $\mathcal{F}$ on X there exists a possibly infinite sequence of sheaves of the form*

$$\cdots \to \bigoplus_{i=1}^{r_i} \mathcal{O}_X(-m_i L) \to \cdots \to \bigoplus_{i=1}^{r_1} \mathcal{O}_X(-m_1 L) \to \mathcal{F} \ ,$$

which is exact off V.

During the project we followed two threads: on the one hand we studied invariants of big line bundles, on the other hand, we started a line of research involving a different concept of partial ampleness. Based on earlier work of Demailly–Peternell–Schneider [12] and Totaro [29], we consider divisors that become ample when restricted to general complete intersection subvarieties. It turns out, that although these divisors are often not even pseudo-effective, the vanishing of their higher cohomology can be controlled.

The sections on Okounkov bodies and volumes of line bundles treat the case of big divisors, Section 3 describes vanishing theorems for partially positive line bundles. We discuss the question of bounded negativity on surfaces in the last section.

1. Okounkov bodies

Among the many invariants that can be associated to big divisors, a collection of convex bodies, the so-called Okounkov bodies, play a central role. They are universal among the numerical invariants of big divisors in that if two big line bundles have the same collection of Okounkov bodies (in a sense that can be made precise), then the two bundles are numerically equivalent [17].

It is in his influential articles [27] and [28], where — building on earlier work of Beilinson and Parshin — Okounkov explains how to associate a convex body $\Delta(D) \subseteq \mathbb{R}^n$ to an ample divisor D on an n-dimensional smooth variety X equipped with a complete flag of subvarieties $Y_{n-1} \supset Y_{n-2} \ldots \supset Y_0$. The convex body $\Delta(D)$ then encodes a lot of information on the asymptotic behaviour of the complete linear system $|D|$. In their excellent paper [25] Lazarsfeld and Mustaţă extend Okounkov's construction to big divisors, which we now briefly summarize, and prove various properties of these convex bodies.

For simplicity, start with a projective variety of dimension n defined over an uncountable algebraically closed field of arbitrary characteristic. Certainly no harm is done if we assume that we work over the complex numbers. Fix a complete flag

$$X = Y_0 \supset Y_1 \supset \ldots \supset Y_{n-1} \supset Y_n = \text{pt}$$

with Y_i an irreducible subvariety of codimension i in X, which is smooth Y_n. For a given big divisor D the choice of the flag determines a rank $n = \dim X$ valuation

$$\nu_{Y_\bullet, D} \colon H^0(X, \mathcal{O}_X(D)) \setminus \{0\} \longrightarrow \mathbb{Z}^n$$
$$s \mapsto \nu(s) \stackrel{\text{def}}{=} (\nu_1(s), \ldots, \nu_n(s)) \ ,$$

where the values of the $\nu_i(s)$'s are defined in the following manner. We set

$$\nu_1(s) \stackrel{\text{def}}{=} \text{ord}_{Y_1}(s) \ .$$

Dividing s by a local equation of Y_1, we obtain a section $\tilde{s_1} \in H^0(X, D - \nu_1(s)Y_1)$ not vanishing identically along Y_1. This way, upon restricting to Y_1, we arrive at a non-zero section

$$s_1 \in H^0(Y_1, (D - \nu_1(s)Y_1)|_{Y_1}) .$$

Then we set

$$\nu_2(s) \stackrel{\text{def}}{=} \text{ord}_{Y_2}(s_1) .$$

Continuing in this fashion, we can define all the integers $\nu_i(s)$. The image of the function $\nu_{Y_\bullet,D}$ in $\mathbb{Z}^n$ is denoted by $v(D)$. With this in hand, we define the *Okounkov body of D with respect to the flag $Y_\bullet$* to be

$$\Delta_{Y_\bullet}(D) \stackrel{\text{def}}{=} \text{ the convex hull of } \bigcup_{m=1}^{\infty} \frac{1}{m} \cdot v(mD) \subseteq \mathbb{R}^n .$$

The construction is analogous to what one meets in toric geometry. In particular, when we take a complete torus-invariant flag, the rational polytope P_D commonly associated to a torus-invariant big divisor D will be a translate of $\Delta_{Y_\bullet}(D)$.

Perhaps one of the simplest examples of an Okounkov body is the one associated to $\mathcal{O}(1)$ on $\mathbb{P}^n$ and a complete flag of linear subspaces. Then the function ν on the sections of $\mathcal{O}(m)$ gives the lexicographic order; the Okounkov body of a divisor in $\mathcal{O}(1)$ turns out to be the standard n-dimensional simplex.

One of the illustrations of the new theory in [25] is the case of surfaces, where they give an explicit description $\Delta(D)$. On smooth projective surfaces Zariski decomposition of divisors provides a tool sufficiently strong for making such a concrete description possible. A complete flag on the surface S consists of a smooth curve C on S, and a point x on C.

Given a surface S equipped with a flag (C, x) and a $\mathbb{Q}$-divisor D, Lazarsfeld and Mustaţă define real numbers ν and μ by setting

$$\nu = \text{ the coefficient of } C \text{ in the negative part of the Zariski decomposition of } D$$

$$\mu = \sup\{t | D - tC \text{ is big }\} .$$

Equivalently, ν is the minimal real number for which C is *not* in the support of the negative part of the Zariski decomposition of $D - \nu C$. As it turns out, the Okounkov body of D lives over the interval $[\nu, \mu]$ described by two functions, $\alpha(t)$ and $\beta(t)$ as follows. Let N_t be the negative part of the Zariski decomposition of $D - tC$ and set $P_t = D - tC - N_t$. Note that although P_t is nef, it is not necessarily effective, though it is linearly equivalent to an effective divisor for all t with $D - tC$ effective. By setting

$$\alpha(t) = \text{ord}_x(N_t|_C), \quad \beta(t) = \text{ord}_x(N_t|_C) + P_t \cdot C .$$

Lazarsfeld and Mustaţă then proceed to prove the following theorem: the Okounkov body $\Delta(D)$ is given by the inequalities

$$\Delta(D) = \{(t, y) \in \mathbb{R}^2 | \nu \le t \le \mu, \alpha(t) \le y \le \beta(t)\} .$$

As a consequence of [3], they conclude in particular that α and β are both piecewise linear and rational on any interval $[\nu, \mu']$ where $\mu' < \mu$.

Our first result in the theory of Okounkov bodies is a sharpening of the Lazarsfeld–Mustaţă statement for surfaces; in fact we show that Okounkov bodies on smooth projective surfaces are always polygons. We also give a complete characterization of the rational convex polygons which can appear as Okounkov bodies of surfaces. The precise statement from [19] goes as follows.

Theorem 1.1. *(Küronya–Lozovanu–Maclean) The Okounkov body of a big divisor D on a smooth projective surface S is a convex polygon with rational slopes.*

A rational polygon $\Delta \subseteq \mathbb{R}^2$ is up to translation the Okounkov body $\Delta(D)$ of a divisor D on some smooth projective surface S equipped with a complete flag (C, x) if and only if the following set of conditions is met.

There exists a rational number $\mu > 0$, and α, β piecewise linear functions on $[0, \mu]$ such that

(1) $\alpha \le \beta$,
(2) β is a convex function,
(3) α is increasing, concave and $\alpha(0) = 0$;

moreover

$$\Delta = \left\{(t, y) \in \mathbb{R}^2 \,|\, 0 \le t \le \mu, \alpha(t) \le y \le \beta(t)\right\} .$$

Again, the proof relies on the methods of [3], we study the variation of Zariski decomposition of the divisors $D - tC$. Fujita's observation that the existence and uniqueness of Zariski decomposition extends pseudo-effective divisors plays a crucial role here. We prove the existence of $\mathbb{R}$-divisors with a given Okounkov body by a direct computation on toric varieties.

It is in general quite difficult to say anything specific about asymptotic invariants of divisors on higher-dimensional varieties. In particular, there is little in the way of regularity that we can expect. It is established in [25] building on a classical example of Cutkosky [9], that there exist big divisors on higher-dimensional varieties with non-polyhedral Okounkov bodies.

Fano varieties, on the other hand, enjoy many favourable properties, which guarantee that all previously known asymptotic invariants behave in a 'rational polyhedral way' on them. In their major breakthrough paper on the finite generation of the canonical ring (see [2], also [6] for a considerably simpler proof) Birkar, Cascini, Hacon, and McKernan also prove that the Cox ring of a Fano variety is finitely generated. Consequently, one could hope that Okounkov bodies associated to divisors on Fano varieties turn out to be rational polytopes. This however is not the case, as the following result of [20] will show.

Theorem 1.2. *(Küronya–Lozovanu–Maclean,* [20]*) There exists a Fano threefold X equipped with a flag $X = Y_0 \supset Y_1 \supset Y_1 \supset Y_2 \supset Y_3$ such that for almost any ample divisor D on X, the Okounkov body of D with respect to the flag $Y_\bullet$ is not a polyhedron.*

Our example is heavily based on a work of Cutkosky [10], where he produces a quartic surface $S \subseteq \mathbb{P}^3$ such that the nef and effective cones of S coincide and are round. The Néron-Severi space $N^1(S)$ of this quartic surfaces is isomorphic to $\mathbb{R}^3$ with the lattice $\mathbb{Z}^3$ and the intersection form $q(x, y, z) = 4x^2 - 4y^2 - 4z^2$. In particular he shows that the divisor class $(1, 0, 0)$ on S corresponds to a very ample divisor class $[L]$ and the projective embedding corresponding to L realizes S as a quartic surface in $\mathbb{P}^3$. We obtain our Fano threefold by blowing up $\mathbb{P}^3$ along an irreducible curve of class $(1, 0, 0)$.

The above construction describes the behaviour of ample divisors for a specific flag, but does not reveal much about generic choices. Our next result shows, that provided one is willing to weaken the positivity hypothesis somewhat, and settle for $-K_X$ big and nef, we can obtain (see [19]) a specimen with strong generic non-regularity. The example in question is a Mori dream space, which means in particular, that it also has a finitely generated Cox ring.

Theorem 1.3. *(Küronya–Lozovanu–Maclean,* [19]*) There exists a three-dimensional Mori dream space Z and an admissible flag $Y_\bullet$ on Z, such that the Okounkov body of a general ample divisor is non-polyhedral and remains so after generic deformations of the flag in its linear equivalence class.*

The construction that leads to this statement is an elaboration of the example is Theorem 1.2.

2. Countability questions and volumes of line bundles

The volume of a divisor on an n-dimensional irreducible projective variety defined as

$$\mathrm{vol}_X(D) \stackrel{\mathrm{def}}{=} \limsup_{m \to \infty} \frac{h^0(X, \mathcal{O}_X(mD))}{m^n/n!}$$

describes the asymptotic rate of growth of the number of global sections as we take higher and higher multiples. Here we will look at volumes of divisors from the point of view of arithmetic. As always, we work over the complex number field.

Along with the stable base locus, the volume is one of the first asymptotic invariants of line bundles that have been studied. It has first appeared in some form in [9], where Cutkosky used the irrationality of the volume of a divisor to show the non-existence of birational Zariski decompositions with rational coefficients. Volumes enjoy many unexpected formal properties, in particular, they give rise to a homogeneous log-concave continuous function from $N^1(X)_\mathbb{R}$ to the non-negative real numbers. For a complete account, the reader is invited to look at [24] and the recent paper [25].

The volume function can be often described in terms of some additional structure on the underlying variety. It can be explicitly determined on toric varieties [16], on surfaces [3], and on abelian varieties and homogeneous spaces for example. In every case, the volume reveals a fair amount of the underlying geometry.

Our main focus here is the multiplicative submonoid of the non-negative real numbers consisting of volumes of integral divisors on irreducible projective varieties. As a starting point, we take the fact that the volume of a divisor with finitely generated section ring is rational. Looking at the low-dimensional situation, an immediate consequence of Zariski decomposition on surfaces gives that every divisor there — even the ones with a non-finitely generated section ring — has rational volume. Conversely, a simple application of Cutkosky's construction provides us with examples to the effect that every non-negative rational number can be realized as the volume of an integral divisor.

Moving on to higher dimensions, we have seen above that the volume of an integral divisor need not be rational, although the example Cutkosky obtains is algebraic, leaving a considerable gap, and the question whether an arbitrary non-negative real number can be realized as the volume of a line bundle. This is the issue that we intend to address from two somewhat complimentary directions.

Theorem 2.1. *(Küronya–Lozovanu–Maclean, [21]) Let $\mathcal{V}$ denote the set of non-negative real numbers that occur as the volume of a line bundle. Then*

(1) $\mathcal{V}$ is countable;

(2) $\mathcal{V}$ contains transcendental elements.

Let us briefly give an idea why the above results hold. The transcendency of volumes of integral divisors is again an application of Cutkosky's principle. In our particular case we consider $\mathcal{O}(1)$ of the projectivization of a rank three vector bundle on the self-product of a general elliptic curve E. We exploit the non-linear shape of the nef cone of the abelian surface $E\times E$. to arrive at the required transcendency of the volume of $\mathcal{O}(1)$. As a by-product of our reasoning one can also see that divisors with transcendental volume show up often and quite naturally in a non-finitely generated setting.

As far as the cardinality of $\mathcal{V}$ is concerned, it is a direct consequence of a much stronger countability result: building on the existence of multigraded Hilbert schemes as proved in [15], we establish the fact that there exist altogether countably many volume functions and ample/nef/big/pseudo-effective cones for all irreducible varieties in all dimensions.

Theorem 2.2. *(Küronya–Lozovanu–Maclean, [21]) Let $V_{\mathbb{Z}} = \mathbb{Z}^\rho$ be a lattice inside the vector space $V_{\mathbb{R}}$. Then there exist countably many closed convex cones $A_i \subseteq V_{\mathbb{R}}$ and functions $f_j : V_{\mathbb{R}} \to \mathbb{R}$ with $i,j\in\mathbb{N}$, so that for any smooth projective variety X of dimension n and Picard number ρ, we can construct an integral linear isomorphism*

$$\rho_X :\ V_{\mathbb{R}} \ \to\ N^1(X)_{\mathbb{R}}$$

with the properties that

$$\rho_X^{-1}(\mathrm{Nef}) \ =\ A_i,\ \textit{and}\ \mathrm{vol}_X \ \circ\ \rho_X \ =\ f_j$$

for some $i,j\in\mathbb{N}$.

The same principle gives the countability of global Okounkov bodies.

Theorem 2.3. *(Küronya–Lozovanu–Maclean, [19]) There exists a countable set of closed convex cones $\Delta_i \subseteq \mathbb{R}^n\times\mathbb{R}^\rho$ with $i\in\mathbb{N}$ with the property that for any smooth, irreducible, projective variety X of dimension n and Picard number ρ, and any admissible flag $Y_\bullet$ on X, there is an integral linear isomorphism*

$$\psi_X : \mathbb{R}^\rho \longrightarrow N^1(X)_{\mathbb{R}}\ ,$$

depending only on X, such that

$$(\mathrm{id}_X \times\psi_X)^{-1}(\Delta_{Y_\bullet}(X) \ =\ \Delta_i$$

for some $i\in\mathbb{N}$.

It must be pointed out that the above countability statements only hold in the complete case. In the non-complete setting, mostly anything satisfying a few formal requirements can occur.

Theorem 2.4. *(Küronya–Lozovanu–Maclean, [21]) Let $K\subseteq\mathbb{R}^\rho_+$ be a closed convex cone with non-empty interior, $f : K\to\mathbb{R}_+$ a continuous function, which is non-zero, homogeneous, and log-concave of degree n in the interior of K. Then there exists a smooth, projective variety X of dimension n and Picard number ρ, a multigraded linear series $W_\bullet$ on X and a positive constant $c>0$ such that* $\mathrm{vol}_{W_\bullet}\equiv c\cdot f$ *on the interior of K. Moreover we have* $\mathrm{supp}(W_\bullet) = K$.

Getting back to the issue of transcendental volumes, it is an interesting fact that the irregular values obtained so far by Cutkosky's construction have all been produced by evaluating integrals of polynomials over algebraic domains. In fact, all volumes computed to date can be put in such a form quite easily, hence they belong to the set of periods, a countable collection of complex numbers that lie between algebraic and arbitrary real, and are studied extensively in various branches of mathematics, including number theory, modular forms, and partial differential equations. An enjoyable account of periods can be found in [18].

To some degree the phenomenon that all known volumes are periods is explained and accounted for by the existence of Okounkov bodies. The actual Okounkov body depends on the choice of an appropriate complete flag of subvarieties, however, an important result of [25] says that

$$\mathrm{vol}_{\mathbb{R}^n}\left(\Delta_{Y_\bullet}(D)\right) = n! \cdot \mathrm{vol}_X(D) ,$$

where the volume on the left-hand side is the Lebesgue measure on $\mathbb{R}^n$, while the one on the right-hand side is the asymptotic rate of growth of the number of global sections of mD. This implies that

$$\mathrm{vol}_X(D) = \frac{1}{n!} \cdot \int_{\Delta_{Y_\bullet}(D)} 1 .$$

Consequently, whenever the Okounkov body of a divisor with respect to a judiciously chosen flag is an algebraic domain, the volume of D will be a period, which indeed happens in all known cases.

This gives rise to the following interesting question: is it true that the volume of a line bundle on a smooth projective variety is always a period?

3. Partial positivity

Much of higher-dimensional algebraic geometry rests on variants of two celebrated theorems about the vanishing of higher cohomology, Serre's vanishing theorem cited above claiming that for an ample line bundle $\mathcal{L}$ and a coherent sheaf $\mathcal{F}$ on a complete scheme X there exists an integer $m_0 = m_0(\mathcal{L}, \mathcal{F})$ for which

$$H^i\left(X, \mathcal{F} \otimes \mathcal{L}^{\otimes m}\right) = 0 \quad \text{for all } i > 0 \text{ and } m \geq m_0 ;$$

and Kodaira vanishing, which says that whenever we have an ample line bundle on a smooth complex projective variety X,

$$H^i\left(X, \mathcal{L} \otimes \omega_X\right) = 0 \quad \text{whenever } i > 0$$

automatically follows. Putting the issue of smoothness aside, the important features of the two theorems are: in Kodaira's statement one has vanishing for the higher cohomology of a given line bundle at the expense of tensoring by the canonical sheaf, while Serre's result has a much wider range of applicability, but one loses control of the actual line bundle the higher cohomology of which vanishes. A powerful interpolation between the two is Fujita's vanishing theorem (for a proof see [13] or [24, Theorem 1.4.35]): let X be a complete scheme, $\mathcal{L}$ an ample line bundle, $\mathcal{F}$ a coherent sheaf on X. Then there exists an integer $m_0 = m_0(\mathcal{L}, \mathcal{F})$ for which

$$H^i\left(X, \mathcal{F} \otimes \mathcal{L}^{\otimes m} \otimes \mathcal{N}\right) = 0$$

for all $i > 0$, $m \geq m_0$, and nef line bundles $\mathcal{N}$ on X (note that m_0 is independent of $\mathcal{N}$).

As we have seen, vanishing theorems classically apply to ample or big and nef line bundles. However, with the recent shift of attention towards big line bundles, it became very important to know what kind of vanishing results big line bundles or even certain non-big ones retain.

It has been known for some time that big line bundles behave in a cohomologically positive way in degrees roughly above the dimension of the stable base locus. An easy asymptotic version of this appeared in [23, Proposition 2.15], while Matsumura [26, Theorem 1.6] gave a partial generalization of the Kawamata–Viehweg vanishing theorem along these lines. In [22] we proved results along the same lines guaranteeing the vanishing of cohomology groups of high degree under various partial positivity conditions.

Inspired by the recent paper [29] of Totaro, we investigate the relationship between ampleness of restrictions of line bundles to general complete intersections and the vanishing properties of higher cohomology groups. This train of thought will eventually lead us to a generalization of Fujita's vanishing theorem to all big line bundles.

Following ideas of Andreotti–Grauert [1] and Demailly–Peternell–Schneider [12], Totaro establishes a very satisfactory theory of line bundles with vanishing cohomology above a certain degree. The concept has various characterizations, which all fall under the heading of q-ampleness. What is of interest to us is the following version, which goes by the name 'naive q-ampleness' in [29]. We call a line bundle L (naively) q-ample on X for a natural number q, if for every coherent sheaf $\mathcal{F}$ on X there exists an integer m_0 depending on $\mathcal{F}$ such that

$$H^i\left(X, \mathcal{F} \otimes \mathcal{O}_X(mL)\right) = 0 \quad \text{for all } i > q \text{ and } m \geq m_0.$$

It is immediate from the definition that 0-ampleness coincides with ampleness, while it is proved in [29, Theorem 10.1] that a divisor is $(n-1)$-ample exactly if it does not lie in the negative of the pseudo-effective cone in the Néron–Severi space. The notion of q-ampleness shares many important properties of traditional ampleness, for example it is open both in families and in the Néron–Severi space. Nevertheless, the behaviour of q-ample divisors remains mysterious in general.

Our immediate motivation comes from unpublished work related to [11], looking for a connection between q-ampleness and geometric invariants describing partial positivity, in particular, amplitude on restrictions to general complete intersection subvarieties. Along these lines, employing ideas from [23] and [11], we can prove strong vanishing theorems for non-ample — oftentimes not even pseudo-effective — divisors. Our proofs all follow the same fundamental principle: positivity of restrictions of line bundles results in partial vanishing of higher cohomology groups. Our first statement is a uniform variant of Serre vanishing.

Theorem 3.1. *(Küronya, [22]) Let X be an irreducible projective variety, L a Cartier divisor, $A_1, \ldots, A_q$ very ample Cartier divisors on X such that $L|_{E_1\cap\cdots\cap E_q}$ is ample for general $E_j \in |A_j|$, $1 \leq j \leq q$. Then for any coherent sheaf $\mathcal{F}$ on X there exists an integer $m(L, A_1, \ldots, A_q, \mathcal{F})$ such that*

$$H^i(X, \mathcal{F} \otimes \mathcal{O}_X(mL + \sum_{j=1}^{q} k_j A_j)) = 0$$

for all $i > q$, $m \geq m(L, A_1, \ldots, A_q, \mathcal{F})$ and $k_1, \ldots, k_q \geq 0$.

Once we know what to aim at, the proof is an induction on codimension using Fujita's vanishing theorem. The basic principle is in fact straightforwards: assume L is a Cartier divisor, A a very ample Cartier divisor on X, for which $L|_E$ is ample for a general element $E \in |A|$. Then from the cohomology long exact sequence associated to

$$0 \to \mathcal{O}_X(mL + (k-1)A) \to \mathcal{O}_X(mL + kA) \to \mathcal{O}_E(mL + kA) \to 0$$

shows that $H^i\left(X, \mathcal{O}_X(mL + (k-1)A)\right) \simeq H^i\left(X, \mathcal{O}_X(mL + kA)\right)$ for all $i \geq 1$ and $k \geq 0$ by Fujita vanishing on E whenever m is sufficiently large. The common isomorphism class of all the groups $H^i\left(X, \mathcal{O}_X(mL + kA)\right)$ for $k \geq 1$ is 0 according to Serre vanishing for the (very) ample divisor A.

In particular, setting $k_1 = \cdots = k_q = 0$ in the Theorem we obtain that L is q-ample. Thus, we recover a slightly weaker version of [12, Theorem 3.4]. As a corollary of this vanishing result, we can observe interesting properties of invariants expressing partial positivity and the inner structure of various cones of divisors in the Néron–Severi space. Here again we go along lines close to [12]; it turns out that sacrificing a certain amount of generality results in radically simplified proofs.

Next, we treat the case of vanishing for adjoint divisors, the setting of Kodaira's vanishing theorem. We obtain a generalization of the theorem of Kawamata and Viehweg.

Theorem 3.2. *(Küronya, [22]) Let X be a smooth projective variety, L a divisor, A a very ample divisor on X. If $L|_{E_1\cap\cdots\cap E_k}$ is big and nef for a general choice of $E_1, \ldots, E_k \in |A|$, then*

$$H^i\left(X, \mathcal{O}_X(K_X + L)\right) = 0 \quad \text{for } i > k.$$

The conditions of Matsumura hold true under our assumptions, hence we recover [26, Theorem 1.6]. The argument that leads to the proof uses homological methods from [23], and the original Kawamata–Viehweg vanishing. Building on the above result, we arrive at our main achievement, a generalization of Fujita's theorem [13] to big divisors.

Theorem 3.3. *(Küronya, [22]) Let X be a complex projective scheme, L a Cartier divisor, $\mathcal{F}$ a coherent sheaf on X. Then there exists a positive integer $m_0(L, \mathcal{F})$ such that*

$$H^i\left(X, \mathcal{F} \otimes \mathcal{O}_X(mL + D)\right) = 0$$

for all $i > \dim \mathbf{B}_+(L)$, $m \geq m_0(L, \mathcal{F})$, and all nef divisors D on X.

Here $\mathbf{B}_+(L)$ denotes the augmented base locus of L; this can be defined as the stable base locus of the $\mathbb{Q}$-Cartier divisor $L - A$ for any sufficiently small ample class A. In essence, we extend Fujita's result to arbitrary big divisors by finding the correct range of vanishing for the higher cohomology.

The proof relies on heavily on properties of divisors that become ample when restricted to general complete intersections, another key ingredient is the resolution result Theorem 0.1.

4. Negativity of curves on algebraic surfaces

Irreducible curves with negative self-intersection have great significance in the theory of algebraic surfaces. They are to a large extent responsible for generating the Mori cone of a surfaces, and as such, play a central role in both classical and modern aspects of the field. While it has been long known that in positive characteristic the Frobenius endomorphism leads quickly to examples of smooth projective surfaces with arbitrarily negative curves (by which we mean that there exists a sequence of irreducible curves C_n on the surface satisfying $(C_n^2) \to -\infty$), it has been a long-standing question whether the same phenomenon can occur over the complex numbers.

In the work [4], we provide various small improvements on existing results. Using the logarithmic Bogomolov–Miyaoka–Yau inequality, we obtain a simple and complete proof of the following lower bound[1]: let X be a smooth projective surface with $\kappa(X) \geq 0$. Then for every irreducible curve $C \subseteq X$ of geometric genus $g(C)$ one has

$$(C^2) \geq c_1^2(X) - 3c_2(X) + 2 - 2g(C) .$$

In a somewhat different direction, we prove the following relation between Seshadri constants and bounded negativity.

Theorem 4.1. *(Küronya et al., [4]) Let X be a smooth projective surface with the property that $(C^2) \geq -b(\widetilde{X})$ for all irreducible curves $C \subseteq \widetilde{X}$ for some non-negative constant $b(\widetilde{X})$, where $\pi : \tilde{X} \to X$ is the blow-up of a point x on X. Then the Seshadri constant $\epsilon(X,x)$ satisfies the lower bound*

$$\epsilon(X,x) \geq \frac{1}{\sqrt{b(\widetilde{X})+1}} .$$

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