Final report on the OTKA project PD 73609, “Stochastic properties of hyperbolic dynamical systems”

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1 Preliminary note

This project was originally planned for 36 months. However, it had to be terminated after 12 months, because I moved abroad for a post-doc position and the rules of OTKA allowed neither the continuation, nor the suspension. Thus I can only report about one year instead of the three years in the plan. As a consequence, results are partial.

The results in this report concern the questions posed in the original research plan. However, in the report I will not follow the structure of the plan – which contained three main sections corresponding to three directions of emphasis – since some results are connected to more than one of these three points. These connections will be indicated. There is one exception: the last result presented (about random trees) has only loose links – hidden in the methods of study – to the planned topics.

2 Results

2.1 Exponential Decay of Correlations in Multi-dimensional Dispersing Billiards

The paper [1] appeared during the project time period, containing results of earlier work connected directly to Part 1 of the plan.

2.2 Complexity of singularities in HD dispersing billiards

This topic belongs to Part 1 of the research plan.

Before submitting the proposal, in [1] we showed exponential decay of correlations for a class of multi-dimensional dispersing billiards. Although this is one of top results in this direction, the proof is conditional: the singularity structure of the system is required to satisfy the so-called *sub-exponential complexity condition*.

To formulate this condition, let S_n denote the singularity set of T^n, the n-th iterate of the billiard map T – i.e. the set of those phase points where at least one of the next n collisions is singular (either tangential, or hits a corner point). This S_n consists of (in our case finitely many) submanifolds of the phase space. Let K_n denote the maximum number of these singularity submanifolds that can meet in a single point. Due to the nature of billiard geometry, S_n cuts a sufficiently small ball in the phase space into at most $K_n + 1$ pieces corresponding to different possible “collision history” sequences of length n. The sub-exponential complexity condition is said to hold if K_n increases slower than any (divergent) exponential function of n. Actually, this can be weakened and replaced by the so-called “sub-expansion complexity” condition, which holds if K_n grows slower than Λ^n where $\Lambda > 1$ is the smallest possible expansion in the unstable directions (of the hyperbolic dynamics).

2.2.1 Example of exponential complexity

In two-dimensional billiards with finite horizon, K_n is known to grow at most linearly, always [4]. Experts generally believed that this is the case in higher dimensions as well. However, shortly after finding the result in [1], we realized that this is not the case. The first result of the project is the construction of a specific 3-dimensional (finite horizon) billiard system where $K_n > 2^{n/2}$ can be rigorously proven. Moreover, the system can be constructed so that Λ is arbitrarily close to one, so even the "sub-expansion complexity" condition fails.

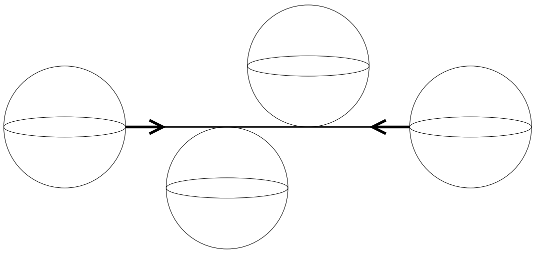

Figure 1: Periodic orbit with exponential complexity. All scatterers are spherical, all centers are in one plane.

The construction is shown in Figure 2.2.1. The four spherical scatterers are positioned in a highly non-typical way so that a periodic orbit is present with several singular collisions. The next figure shows how exponential complexity appears in this setting. The phenomenon is very similar to the well-understood (see e.g. [11]) case of the corridor in a 2-*dimensional, infinite horizon* dispersing billiard, where exponentially many collision sequences are possible.

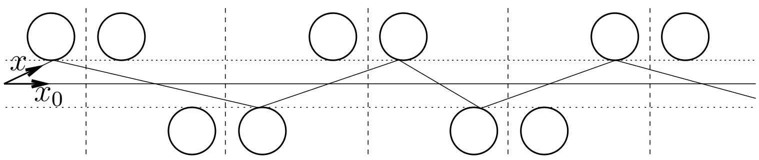

Figure 2: Corridor in an infinite horizon planar billiard with many possible collision history sequences near a reference trajectory.

The proof has been completed during the project period. The paper [2] presenting the result is in preparation.

2.2.2 Typicality of sub-exponential complexity

The billiard system presented in the previous result is very particular. Many conditions have to be met simultaneously, so we do not expect the observed phenomenon to be typical. On the contrary, we conjecture that finite horizon dispersing billiard systems with sub-exponential complexity are typical in every reasonable sense of typicality. Indeed, a simple heuristic dimension-counting argument suggests that the complexity K_n is typically not just sub-exponential, but even bounded.

Reasonable notions of typicality include measure-theoretic typicality of parameters of algebraic manifolds in some parameter space, or topological typicality of sets of smooth manifolds in some C^k topology. I managed to construct many counter-examples indicating why a purely algebraic approach to a proof can not work. By now we have a good understanding of how and why the hyperbolicity of the dynamics should be exploited, but no rigorous proof could be created.

However, there is an important positive partial result for the topological typicality. Namely, I have proven that *in two dimensions* the complexity is bounded (by 3) for a C^3-typical set of scatterer curves. The proof is done via a fairly explicit (quite non-trivial) construction of a sequence of C^3-small perturbations that convert any configuration into a nearby element of a $\mathcal{G}_\delta$ set where every phase point is at most twice singular. Unfortunately, this does not resolve the utmost important question about higher dimensions. Work is being done to generalize the construction, but there are serious technical difficulties. The way of publication of the 2-dimensional result will depend on the outcome of these efforts.

2.3 Reliability of numerical simulations in hyperbolic systems

This topic belongs to Part 3 of the research plan.

It is the nature of hyperbolic dynamical systems that the true trajectory of a particular phase point is usually hopeless to follow by numerical calculation. However, when one studies statistical properties, this is not the only difficulty, and not even the main difficulty. Indeed, in many interesting situations, the numerically calculated sequence of phase points *will* be close to *some true trajectory* – although not the true trajectory of the initial phase point, but the trajectory of another, nearby one. (This is the essence of so-called "shadowing lemmas".) This would suffice for statistical properties: we are not interested in that particular trajectory anyway: we are interested in *typical* trajectories. The main problem is that in general there is no guarantee that the trajectory we see is typical in any sense. The most transparent example is the dyadic map $x \mapsto 2x \pmod 1$ on the $[0, 1]$ interval. If one uses the usual binary arithmetic to represent numbers in [0,1], then the simulation follows the trajectories of all (represented) points with no error at all. However, these (finitely many) points all exhibit highly non-typical behaviour – namely they are mapped to 0 in finitely many steps, unlike Lebesgue-almost every point of the interval. A possible explanation of the phenomenon is that – due to the expansivity of the map – not all points in the discretization can be obtained as images by the map. Gábor Domokos and Domokos Szász suggested a remarkably simple method to overcome this difficulty. Roughly: in every step of the iteration, a small random perturbation is added to "smuggle back" the phase points lost because of the expansion. In [5] they could prove rigorously that this method can be used to find the physical measure for the simplest piecewise linear expanding interval maps.

With Róbert Sepsi we studied the usability of such randomized iteration algorithms in an empirical way, by performing simulation on well-understood systems not covered by the result of [5]. We were particularly interested in systems with big expansion at certain points, since the size of the random perturbation of [5] is proportional to the derivative. Our experience can be summarized as follows:

- In practice there is no need to use big random perturbations where the derivative (some derivative) is big. Indeed, moderate random perturbations (choosing from no more than, say, 5 possible points in one dimension, 25 in two,) give excellent results for all practical purposes.
- Using such moderate perturbations has – beside the simplicity of programming – another huge advantage. If the studied system is not (known to be) ergodic, applying big random perturbations often leads to highly misleading simulation results. Namely, the simulation can pass from one ergodic component to the other, making the system appear as if it were ergodic. If perturbations are kept moderate, then a sufficiently dense discretization of the phase space can make the probability of such transitions negligible, even in a long simulation.

Details and further results are in [10].

2.4 Heat conduction in lattices of coupled hyperbolic systems

Preliminary: the Gaspard-Gilbert heat conduction model

In [6] Gaspard and Gilbert presented the following ingenious heat conduction model: Consider the situation in Figure 3. The striped circles are fixed walls, while the small disks move according to billiard dynamics. The geometric parameters are chosen so that the disks remain localized (they can't escape from their "cages"), but they can collide – and exchange energy – with their neighbours. It is especially interesting to study the "low frequency" limit, when the geometry is chosen so that the "energy exchange" collisions are very rare compared to the collisions with the fixed walls. This allows the disks to "equilibrate"

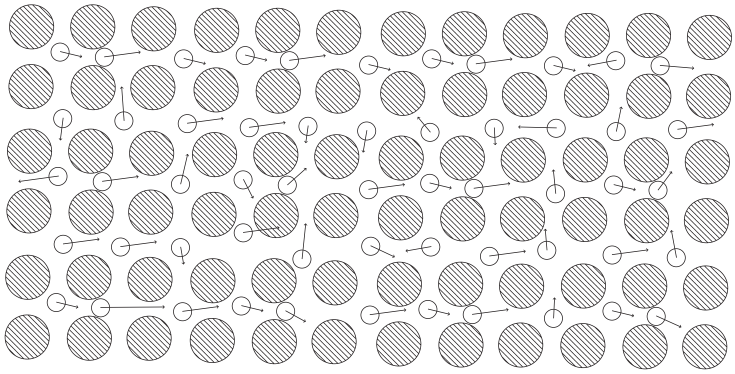

Figure 3: Gaspard-Gilbert heat conduction model

between any two energy exchanges, and leads – in the appropriate scaling limit – to a Markov process that governs the energies stored at lattice cites: that is, an interacting particle system. The hydrodynamic limit behaviour of this interacting particle system can in turn be studied with probabilistic methods, "forgetting" the dynamical system behind.

This work (and more) was done at a heuristic level in [6].

2.4.1 Gaspard-Gilbert model with weak interaction forces

Despite the beauty and simplicity of the above model, very little can be said about it rigorously. Without going into details, the main problem is that this "low frequency" limit is not a "weak coupling" limit: although energy exchange collisions are rare, the energies exchanged at such occasions are big, and the particles get big "kicks" which possibly bring them far from their local "equilibrium" state. To overcome this difficulty, it is natural to consider a modified – and equally interesting model. Here the collision between the moving disks is ruled out, but neighbouring disks interact via some conservative force, which is set to be small in an appropriate scaling limit. This can now be legitimately called the "weak coupling" limit.

A big part of my work during the project period was the study of this model with both rigorous and heuristic tools as well as numerical methods. I give some details of the results, since at the moment only the simulation work is published electronically.

1. For finitely many particles, the weak interaction limit leads to an interacting particle system. The energies E^i satisfy the following system of coupled stochastic differential equations: for every lattice site i
$$\mathrm{d}E_t^i = \sum_j \left(b(E_t^i, E_t^j)\,\mathrm{d}t + \sigma(E_t^i, E_t^j)\,\mathrm{d}B_t^{ij} \right), \tag{1}$$
where the sum runs over all neighbours j of i, and the B_t^{ij} are Wiener processes which are independent for different edges, but $B_t^{ij} = -B_t^{ji}$. In words:
    - On every edge of the lattice there sit independent Wiener processes governing the energy transfer through the edge,
    - the drift b and the diffusion coefficient σ of the transfer through the edge depends only on the energies at the sites connected.

   The proof is based on a detailed study of correlation decay in the two-particle system, and thus belongs to Part 1 of the research plan.

2. The functions b and σ feature several important symmetries:
    - $b(x, y) = -b(y, x)$ and $\sigma(x, y) = \sigma(y, x)$, which corresponds to conservation of energy.

- b can be expressed in terms of σ, which corresponds to the universality of the invariant measure, inherited from the invariant (Liouville) measure of the Hamiltonian system we started with.
- σ is homogeneous in the total energy involved: $\sigma(Ex, Ey) = E^{1/4}\sigma(x, y)$, which is extremely useful in the study of the hydrodynamic limit.

3. The diffusion coefficient σ can be obtained as a time autocorrelation (Green-Kubo) integral. Its form is utmost important concerning the behaviour of the solutions of the stochastic differential equation. It was investigated by numerical simulation of the 4-dimensional cylindrical billiard formed by two neighbouring disks, which belongs to Part 3 of the research plan. The simulation is published electronically [12]. An example is shown in Figure 3.

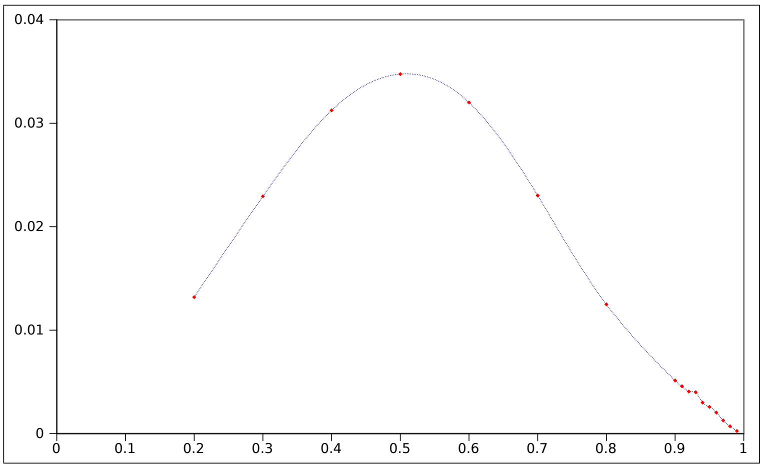

Figure 4: $\sigma(x, 1-x)$ as a function of x when the interaction force between neighbours is harmonic. The behaviour near zero is not simulated in detail, since the function is known to be symmetric.

4. The numerics suggests convincingly that σ vanishes linearly as one of the particle energies approaches zero. This implies that no boundary conditions at zero are needed for the stochastic differential equation (1): the solution remains positive automatically. In fact, near vanishing particle energy the system behaves exactly like the square Bessel process. This is especially important since the method of proof for Result 1 breaks down when one of the particles stops. At the moment we only have the numerical indication that this is not a serious weakness.

5. Strong heuristic arguments let us believe that the interacting particle system described by (1) leads to normal heat conduction in the hydrodynamic limit. Accepting that, the symmetries in Result 2 fix the temperature dependence of the thermal conductivity up to a constant. Namely, we get

$$D(T) = const\, T^{-3/2}. \tag{2}$$

Summarizing the results: a mixture of rigorous, heuristic and numerical methods leads to a full picture of heat conduction in the weak coupling limit of the model. The thermal conductivity (2) is in surprisingly good agreement with experimental data on insulators and basic semiconductors at high temperature (see e.g. [7], section 3.2.3). This is in big contrast with the original Gaspard-Gilbert model, where the heat conductivity increases with temperature.

2.4.2 Further weakly coupled lattices of hyperbolic systems

The main difficulty in handling the weak coupling limit in billiards is that the theory of mixing is based on a complicated and detailed analysis of the unstable manifold evolution in the specific situation. Treating the continuous time flow is especially troublesome. So it is tempting to consider simpler models, like lattices of coupled expanding maps, where mixing is easier to handle with the transfer operator technique discussed in Part 2 of the research plan. In some cases the result corresponding to the above equation (1) can be proven with this technique.

2.5 Entropy and Hausdorff Dimension in Random Growing Trees

This result has only loose links to the research plan of the project. Still I mention it briefly, since it was done during the time of the OTKA support.

Consider a family tree stemming from a single individual in the following way:

- At time 0 there is only one member, called the "root".
- At time 1 a new member joins and becomes the daughter of the root.
- At every further integer time moment a new individual arrives, and chooses a mother from the already existing family members *randomly*, giving possibly different chance to different individuals, based on the number of children they already have.

Such a process is called a *preferential attachment model*, and is a popular tool to describe the growth of social or informatics networks, reflecting phenomena like "if you have more friends, you find new friends easier". The most famous example is the Barabási-Albert model [3].

A natural way to describe the long-term structure of the tree is through a *random measure* on the set of leaves (infinite branches) of the tree: the weight given to the sub-family stemming from some individual will simply be the proportion of the number of sub-family members to the full population after a long time: this proportion is known to converge almost surely under mild assumptions [9].

In [8] we show that the Hausdorff dimension of this random measure is constant with probability one. Moreover, the relation of the Haudorff dimension to the entropy is shown, and an explicit formula is given in terms of the rule of preferential attachment. The present paper covers the case when any mother can only have finitely many children, but otherwise the preferential attachment rule is arbitrary. At the heart of the proof stand a simple expanding map and an ergodic dynamical system (arising from a Markov chain).

3 Summary

The work of this one year concentrated on two of the three parts of the research plan. No serious new result was reached in Part 2, although the techniques discussed there were applied in some cases. In the other two parts, important work has been done and interesting results have been reached. The importance of probabilistic problems, like stochastic differential equations, that I encountered during the study of the deterministic models was also higher than expected. In my opinion, the results I can account for contribute a good portion of the plan for this time period. Especially the entirely new results of Section 2.4.1 are serious achievements in an area that is in the center of international attention.

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