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The methods and intuition of physicists, in particular, of statistical physicists are utmost successful and absolutely necessary to gain insight into the laws of nature. At the same time, the rigorous understanding of arguments of physicists provide a deeper perception of laws and lead to certitude. In several directions of statistical physics, billiard models - specially hard ball ones - have become most successful in this direction. It is sufficient to mention the description of diffusion, superdiffusion, Brownian type motions, thermostated systems, motion in force fields, anomalous currents, ... Recently much interest has been focused on heat conduction and, in particular, on energy transfer. For instance, the model [GaGi08] figuring in our project, is closely related to condensed matter physics [CMR] or nanostructures [CMBSC]. Billiard type models and related random walk problems have been in the center of our research in parts I and III.

I Hard ball systems, hyperbolic billiards & Lorentz process

I/A

Multidimensional Hyperbolic Billiards

When considering physically motivated chaotic dynamical models with finite size, statistical properties of hyperbolic billiards, including hard ball systems, are among the most interesting and most studied subjects. Within that, especially the multi-dimensional case is the specialty of the Budapest school. Presently the focus is on strong statistical properties, but in many cases even ergodicity is still an issue.

Ergodicity and correlation decay

We generalized the theorems concerning strong statistical properties of planar dispersing billiards to the multi-dimensional setting. The involved solution of this problem, which was in the focus of attention, brought wide international recognition.

Since local ergodicity for the most interesting hyperbolic billiards was only known in the case of algebraic scatterers [BChSzT02], the study of strong statistical properties has necessarily lead – as a kind of side-result – to considering the problem of ergodicity.

In [BT08] we prove a long-standing open conjecture: correlations for Hölder continuous observables decay exponentially in multi-dimensional finite-horizon dispersing billiards; under the so-called sub-exponential complexity condition on the singularity set. The main ingredients of our argument are a Young tower construction and the proof of related growth lemmas which require a detailed geometric analysis of the system.

This result attracted exceptional international attention. First, the authors could present the result in the form of a minicourse at the Ervin Schrödinger Institute in Vienna in 2008, during the semester on Hyperbolic Dynamical Systems. Second, the paper [BT08] containing the result brought the authors the Annales Henri Poincaré Best Paper Award for 2008 with the possibilty to present the method at the ICMP 2012 in Aalborg.

In [BaBaT] a Hopf construction is performed and thus a local ergodicity theorem is proved for hyperbolic systems with singularities under a natural set of assumptions, the most important of which are appropriately formulated conditions on the growth of unstable manifolds. As an application we prove the ergodicity of multidimensional dispersing billiards with sub-exponential complexity. The papers [BaBaT] and [BT08] are strongly interrelated.

In [BT12] a multidimensional dispersing billiard configuration is presented and it is shown that, in the vicinity of a periodic orbit of the configuration, the complexity of the singularity set grows exponentially. This counterexample is surprising and is in contrast with the results of [BT08] which concern dispersing billiards with sub-exponentially growing complexity.

Open problems

We have initiated new directions of research. Several of our initiatives have already become successful and got, at least, partial solutions.

The invited problem paper in Nonlinearity, [Sz08] presented some old and new, important and - in our view - nice questions of the theory: the complexity problem of higher dimensional dispersing billiards and its important applications, the problem of ergodicity of hard ball systems in a box, the Erdőtarcsa conjecture of Simányi and Szász, the question on the joint motion of two Lorentz discs (this – for a 2D paradigm walk model – was answered in [P-GySz12]) and the problem of diffusion for a quasi-periodic Lorentz process, for instance the Penrose-Lorentz process (beautiful results on this question were obtained by [KS13] and [T10])).

Algebro-geometric approach for hard ball systems

We have also surveyed a direction of research where the Budapest school had reached sensational breakthroughs earlier.

For the study of hard ball systems, the algebro-geometric approach appeared in 1999 — in a sense surprisingly but quite efficiently — for proving the hyperbolicity of typical systems (see [SiSz99]). By subsequent improvements of this breakthrough method finally Simányi [S13] could verify the celebrated Boltzmann–Sinai ergodic hypothesis. More than that, at present, the best form of the local ergodicity theorem for semi-dispersing billiards, [BChSzT02] also uses algebraic methods (and the algebraicity condition on the scatterers). The goal of the survey work [Sz08] was to summarize the essential steps of the algebro-geometric approach by assuming and using possibly minimum information about hard ball systems. The paper also described the status of ergodicity proofs for cylindrical billiards, a class – introduced by Szász, [Sz93] – generalizing hard ball systems.

I/B

Planar Lorentz Processes. Related Random Walk Problems

Inhomogeneities in time and in space

With the spectacular successes of the theory of Lorentz processes, periodic in time and homogeneous in space, several physically interesting problems became tractable where there are local impurities in the model ([DSzV08, DSzV09]) or the dynamics is time-inhomogeneous ([NaSzV]).

Let us modify the scatterer configuration of a planar, finite-horizon Lorentz process in a bounded domain. Sinai asked in 1981 whether, for the diffusively scaled variant of the modified process, convergence to Brownian motion still holds. The main result of the work [DSzV09] answered Sinai's question in the affirmative (other types of interesting local perturbations were also investigated). The arguments start from those of [SzT81]. The proofs combined Stroock-Varadhan's martingale method with result of the preparatory work [DSzV08]. In this latter paper first return and first hitting times, local times and first intersection times were studied for planar finite horizon periodic Lorentz processes. Their asymptotic behavior was analogous to the asymptotic behavior of the same quantities for the 2D simple symmetric random walks (cf. classical results of Darling-Kac, 1957 and of Erdős-Taylor, 1960).

In [NaSz] we ascertain the scaling limit of a periodic Lorentz process in a strip with an almost reflecting wall at the origin. The limiting process - which we call a quasi reflected Brownian motion - is a Markov but non-strongly Markov process and is interesting in its own right.

In [NaSzV] we prove central limit theorem for certain time dependent dynamics. We give sufficient conditions for the variance to go to infinity in general settings (i.e. the applied maps are not necessarily close to each other). It is still an open problem how to apply the method to higher dimensional models.

Random walk problems motivated by dynamical problems

Often, because of the technical difficulties of the question, one first studies the analogous problem for random walk models. In this case the random walk ideas even became repeatedly useful in the methods used for the solution of the dynamical task (as, for instance, in local perturbations of the Lorentz process, see above).

One of the most interesting questions regarding the range of random walks is the asymptotic number of visited points. In [Na1] we extend the classical results of Dvoretzky and Erdős to the range of random walks with internal states. They can be thought of as a toy model of Lorentz processes.

In the work [P-GySz12] (with a preliminary version [P-GySzT10]) we solved a question raised in [Sz08] for a paradigm model (some technical tools were elaborated in the works [P-GySz11, P-GySz13]). The result also suggests the answer for the original question.

In [PSz10] we could treat the limiting properties of heavy-tailed random walks with local impurities.

In [Na2] we prove interesting recurrence properties of a heavy-tailed random walk. Its one step distribution is the same as that of Lorentz processes with infinite horizon, where analogous properties are conjectured. Our results contribute to a better understanding of the path properties of Lorentz processes.

Planar hyperbolic billiards and related models

Two-dimensional models are much better understood than their multidimensional counterparts, thus our research focused on more complex phenomena and finer statistical properties. One of the main goals is to understand intermittent phenomena: non-uniformly hyperbolic effects may slow down the rates of mixing and result in non-standard limit laws, as in dispersing billiards with cusps. In convex billiards, which can be regarded as perturbations of stadia, the intermittent effects are so strong that even the question of ergodicity vs. KAM islands is a relevant issue. In addition to billiard systems, our investigation was extended to another class of physically relevant models, the systems of falling balls, where the rate of mixing is proved to be polynomial. Further details are given below on our results on these problems.

System of Two Falling Balls.

The system of two falling balls, introduced by Wojtkowski, describes the motion of two point particles under a constant gravity force, located along a vertical half-line and colliding elastically with each other and the floor. It follows from previous work by Liverani and Wojtkowski that the system is mixing if and only if the upper ball is heavier. In [BaBoN] we prove that intermittent phenomena – consecutive bounces of the lower ball on the floor before colliding with the upper one – results in polynomial mixing; modulo logarithmic factors, correlations decay at a $1/n^2$ rate. In accordance with the summable decay rate, the central limit theorem is also proven.

Convex billiard tables

In [BHHS] the Benettin-Strelcyn oval billiard construction is generalized and the resulting two-parameter family is investigated by a combination of numerical (Lyapunov-weighted dynamics) and analytic (evolution of curves in the phase space) arguments. Our most important observations concern the formulation of a conjecture on a new class of ergodic convex billiard tables, the significance of which is related to the lack of the defocusing mechanism which is the key effect behind the ergodicity of the previously known examples.

Planar Dispersing Billiards with Cusps.

When two boundary components in a dispersing billiard table touch tangentially, a cusp is formulated. In the vicinity of the cusp the billiard particle performs an unbounded number of successive collisions which slows down the rate of mixing. It had been proven previously by Chernov and Markarian that correlations decay at a $1/n$ rate in dispersing billiard *maps* with cusps. In [BM08], however, we prove that the rate of mixing is much faster for the billiard *flow*: correlations in continuous time decay faster than any polynomial. As a byproduct of our argument analogous results are obtained for some other classes of planar hyperbolic billiards (e.g. Bunimovich flowers).

[BaChD] considers billiard maps with cusps from the perspective of statistical limit laws. It is shown that, in accordance with the $1/n$ decay rate, typical Hölder observables satisfy a non-standard limit theorem: the superdiffusive $\sqrt{\frac{n}{\log n}}$ scaling is needed for convergence to the Gaussian law. The analogous weak invariance principle is also proven, and discussion of standard (diffusive) limit laws in the degenerate case is included as well.

II Parameter dependence of attractors and invariant measures

II/A Projection, slices and algebraic difference

Palis and Takens [PT] examined the set of parameters in the case of parameterized family of diffeomorphisms, for which there exist homoclinic points. This is our motivation for the examination of algebraic difference of two random Cantor sets. We want to give a sufficient condition that the difference contains internal points. If the difference contains an interval then the set of parameters, which was examined by Palis and Takens, contains an interval, as well. That is, the set of parameters is "large".

Algebraic difference of two random Cantor sets

Palis conjectured that if the sum of the Hausdorff dimensions of two typical dynamically defined Cantor set C_1 and C_2 is greater than one then their arithmetic difference set $C_2 - C_1$ should contain an interval. In [DSSz] this Conjecture was verified for the so-called Larsson's family of random Cantor sets.

In [MSS] two fractal percolation random Cantor sets are constructed on the line, for which the arithmetic difference set has strictly positive Lebesgue measure, but it contains no interval. Palis conjectures that this is not possible for typical dynamically defined Cantor sets.

Projection and slices

It is a well known phenomenon that the algebraic difference of two sets is the projection of their direct product with angle $\pi/4$. During the examination of the algebraic difference it is important to understand the nature of the projection with fix angles.

A Theorem of Marstrand asserts that for a set $E \subset \mathbb{R}$ with $\dim_{\mathrm{H}} E < 1$ for Lebesgue almost all α, the α-angled orthogonal projection of E has the same dimension as E. In

[RS] this dimension preservation property is verified for ALL rather than for almost all projections, in the case of Mandelbrot percolations on the plane.

It was proved by Marstrand 1954 that the Hausdorff dimension of almost all slices (in the most natural sense) of an $s > 1$-set on the plane is $s - 1$. We proved in [MS] that for a Sierpiński carpet and for any angle α having rational slope, for Lebesgue almost all α-slices have dimension less than $s-1$. We state also a dimension conservation phenomena between the box dimension of the slices and the local dimension of the projected natural invariant measure.

We extended the previous result for the Sierpiński gasket in [BFS]. Let F be the right angled Sierpiński gasket. Fix an angle α with rational slope. Let ν be the natural measure on F. We prove that

- a ν-typical α-slice has dimension greater than $s - 1$,
- an α-slice through a Lebesgue typical point of the unit square has dimension smaller than $s - 1$.

We also give a non-complete multifractal analysis of slices.

II/B Hausdorff dimension of hyperbolic attractors

The dimension theory of hyperbolic attractors in higher dimension is far away from well understood. The simplest hyperbolic dynamical systems are the self-affine iterated function systems. We cannot calculate the dimension even in the case when the affine transformations are diagonal matrices.

Overlapping self-similar sets

The overlapping self-similar sets on the line play important role in the study of diagonally self-affine sets. If we take the projection of a diagonally self-affine set onto the coordinate axis, the projected set will be self-similar.

In [B3] and [B4] we study two families of self-similar IFSs with non-distinct fixed points. In both of the cases we assume that the images of the convex hull of the attractor are overlapping only for the functions which share the same fixed point. In [B3] we suppose that every fixed point belongs to at most two functions. In [B4] we assume that there are exactly two different fixed points but a fixed point belongs to arbitrary many functions. In both of the papers we calculate the Hausdorff and box dimension for almost every contracting parameters. Moreover, in [B3] we prove that the proper dimensional Hausdorff measure of the attractor is zero. The first result for the dimension of iterated functions systems with fixed point correspondence was [B2], which is generalized in [B3] and [B4].

Non-conformal sets

One of the most important tool of the field of the dimension theory of non-conformal sets is the sub-additive pressure, which was defined by K. Falconer [F]. Unfortunately, we know very little about sub-additive pressure itself. In [B1] we verify a so-called insensitivity property in a special case of non-conformal IFS consists of maps with lower triangular derivative matrices.

Moreover, in [B4] we consider a special family of diagonally self-affine sets which is called the generalized 4-corner set. We determined the box dimension of this set for Lebesgue typical contracting parameters. The calculation of the dimension uses the result of overlapping self-similar sets.

Stationary measures

In [BPS] we study the Blackwell- and Furstenberg-measures, which play an important role in information theory and the study of Lyapunov exponents. For the Blackwell-measure we determine parameter domains of singularity and give upper bounds for the Hausdorff dimension. For the Furstenberg-measure, we establish absolute continuity for some parameter values. Our method is to analyze linear fractional iterated function schemes which are contracting on average, have no separation properties and, in the case of the Blackwell-measure, have place dependent probabilities.

In [BP] we investigate some properties of the invariant measure of random IFSs. We consider a conformal iterated function system on the real line and add an ε small uniformly distributed i.i.d error at each iteration. We prove that under some assumption the invariant density is in L^2 and the L^2-norm does not grow faster than $1/\sqrt{\varepsilon}$, as ε vanishes.

III Parameter dependence of invariant measures of physically motivated systems

Modeling heat conduction phenomena by one dimensional chains of dynamical systems

Having reached important progress in the study of diffusion, superdiffusion, correlation decay, etc., in the last decade several leading schools of mathematicians and physicists working on dynamical systems turned their attention toward understanding heat conduction. Our research has also reached very strong results in various directions.

The result of [GKhSz12] is only slightly related to topic III. It could not be planned since it is a major step in a program - suggested by Gaspard and Gilbert, [GaGi08] in 2008, only - for establishing Fourier's law of heat conduction. Rather than describing their program in detail let us mention that in the second part of their approach: hydrodynamical limit transition should be established for Markov jump processes of energies of an N-chain with particular n. n. interactions depending on the dimension of the interacting physical balls. Our work

1. introduces a class of general Markov jump processes containing those of the previous authors;

2. describes all reversible invariant probability measures,
3. and under some additional conditions proves a $\frac{const.}{N^2}$ lower bound for the spectral gap, where N is the size of the chain (Sasada in her invited talk on IAMP12 and in her work arXiv:1305.4066 could remove our conditions)

For the chain studied in [BLY] the elementary cells (the systems at the vertices) perform integrable motion and the regular dynamics are only broken by the stochastic heat baths located at the two endpoints of the chain. In equilibrium – when the two bath distributions coincide – an explicit description of the invariant measure is given; while, more importantly, its uniqueness is proven out of equilibrium. This requires a careful study on how the randomness of the baths penetrates all degrees of freedom in the system. Further features of this model, the transport properties in particular, are studied by numerical methods.

Closing remarks. Beside the Best Paper Award mentioned above, we have been invited to several prestigious conferences, schools. In particular, D. Szász was an invited speaker on the International Congress of Mathematical Physics held in Prague in 2009. From among the junior participants, B. Balázs got a 1-year postdoc position in Banach International Center, Warsaw, Zs. Pajor-Gyulay a PhD scholarship at U. of Maryland, College Park, A. Némedy-Varga at BUTE and P. Nándori a 2+1 year postdoc position at New York University.

A new great development in the work of our group is the joint Dynamical Seminar series with the dynamical system group of Vienna University. It is an additional attractive force for visitors from all over the world
(cf. http://mat.univie.ac.at/ zweimueller/BudWiSer/Budwiser.html

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