Report about

Inequalities in information theory

The subject is mathematical, but the problems have been typically motivated by quantum theory. There is a good research relation with several foreign people. The PhD of young people has been relevant.

Relative entropy. The subject started in probability theory, but it was extended also to the non-commutative situation motivated by quantum theory. The relevant subjects are well-described in the new book [1]. The classical relative entropy was extended to fix a function f by Imre Csiszár and the quantum version was introduced by Dénes Petz. The formulations are quasi-entropy or f-divergence. An overview is in the paper [2] which was motivated by the conference for the 70th birthday of Imre Csiszár. There are several new results in the recent paper [3]. The monotonicity under a positive mapping is rather standard, but the case of equality is a good and important subject. The classical book of Imre Csiszár appeared recently again [4], but he had also a quantum motivation in the paper [5], which included the proof of a conjecture of a Hungarian physicist.

Riemannian geometry and quantum Fisher information. Classical Fisher information is from 1920's and the geometric description is in the 1950's. The quantum situation is relatively recent, the Fisher information is not unique and there are several different subjects. An overview is in the paper [6] and very hard study in the possible Riemannian geometry is a matrix analysis subject [7]. The relevance of quantum Fisher information to uncertainty principle was studied in the papers [9, 8].

Quantum state discrimination. The problem of asymptotic binary state discrimination (i.e., identifying the state of a quantum system with reasonable certainty within a set of two alternatives) can be studied in different scenarios, depending on the relative importance associated to the competing alternatives. The settings of the so-called Chernoff and Hoeffding bounds had been an open problem for quite a while, and have been solved recently in the i.i.d. (independent and identically distributed) setting, based on two fundamental inequalities; one relating the Rényi relative entropies of the states to their trace-norm distance [10], and the other relating the trace-norm distance of the quantum states to the trace-norm distance of a pair of associated classical probability distributions [11]. We used these inequalities to extend the above results to the problem of discriminating physically relevant correlated states on quantum lattice systems, including the Gibbs states of translation-invariant finite-range interactions, finitely correlated states [12, 13], temperature states of non-interacting fermionic [14] and bosonic [15] systems as well as the discrimination problem of i.i.d. states under group covariant measurements [16]. We showed that in these setups the Rényi relative entropies not only determine the optimal exponential decay rates of the relevant error probabilities (through the derived quantities of the so-called Chernoff and Hoeffding distances), but the existence and smoothness of the averaged Rényi relative entropies also provides a sufficient condition for the equality of the optimal decay rates and the corresponding operator distinguishability measures for generally correlated states.

We also applied state discrimination results to derive asymptotically sharp lower [17] and upper [18] bounds on the classical information transmission capacity for a single use of a quantum channel. These bounds yield the well-known Holevo-Schumacher-Westmoreland coding result in the asymptotics for i.i.d. channels, but they also provide useful bounds on the capacity in the more realistic scenario where a channel can only be used finitely many times, and the consecutive uses are possibly correlated.

Finally, we showed [3] that if a quantum channel preserves the pairwise distinguishability of members of a set of quantum states, as measured by the Chernoff and the Hoeffding distances, then the channel has a canonical inverse on that set of states (i.e., the channel is sufficient for that set of states, or in other words, the error represented by the channel map can be corrected by a canonical correction channel). This was based on the well-known reversibility result of Petz for Rényi relative entropies [19], which we also generalized for a large class of quantum f-divergences in [3].

Gaussian Markov property. The classical Markov property is in probability theory and it can be formulated by a special property of the entropy. This and typically the vector-valued situation was studied in the paper [20] which was the preliminary in the quantum CCR situation [21, 22]. The CCR example is related to a special function $\gamma(x) = x \log x - (x+1)\log(x+1)$. Motivated by this the strongly subadditive function have been studied, there is a sufficient condition [23].

The members participated in several conferences. The PHD procedure of C. Ghinea, J. Pitrik, T. Baier and A. Szántó was a good part of the work.

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