Final Report

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2021

Uncertainty is a prevailing phenomenon in social and natural situations. If in a situation the state of the world is not known the decision maker faces uncertainty. We distinguish three types of uncertainty: (1) risk, when the uncertainty can be described with a probability distribution; (2) ambiguity, when the probability of some events is not known precisely; and (3) unawareness, when even some events are not known, more precisely, not known that not known and so on by the decision maker. In this project we have considered ambiguity.

Ambiguity is a senior notion in decision theory and in economics general. Ambiguity – more precisely the attitude towards it – can explain important paradoxes in human decision making, see e.g. the Ellsberg paradox (Ellsberg, 1961). The literature of ambiguity is very broad it includes Schmeidler (1989); Gilboa and Schmeidler (1989); Ghirardato and Marinacci (2002); Klibanoff et al (2005); Marinacci and Montucchio (2006); Maccheroni et al (2006); Cerreia-Vioglio et al (2011); Lehrer (2012); Gilboa and Marinacci (2016); Li et al (2018), among others; for an overview of the notion and literature of ambiguity see Machina and Siniscalchi (2014).

It is quite widespread among the above mentioned papers that ambiguity does not appear in those as a primitive of the model, but it is encoded in the preferences of the decision maker. This situation similar to the one of risk, where in von Neumann and Morgenstern (1944) the risk is a primitive of the model, hence it is objective – it does not depend on the decision maker, while in Savage (1954) the considered uncertainty is encoded in the preferences of the decision maker, hence it is subjective – it depends on the decision maker indeed (Anscombe and Aumann (1963) make this distinction – objective vs. subjective – explicit, in Anscombe and Aumann (1963) both types of risk (roulette vs. horse race) are considered). In other words, typically ambiguity is subjective in the models of the above mentioned papers.

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In this project we have followed the multiple prior approach to model ambiguity. The subject of multiple priors has been proposed as maximin expected utility in Wald (1950). From then on it has been discussed also in Hurwicz (1951), Arrow (1951) and Luce and Raiffa (1957). Gärdenfors and Sahlin (1982) made the connection of multiple priors to Ellsberg's paradox (Ellsberg, 1961). Gilboa and Schmeidler (1989) gave an axiomatization of this approach. Particularly, we consider the case when the set of priors is given by a belief function (Dempster, 1967; Shafer, 1976; Jaffray, 1992).

In some situations ambiguity can be used to gain strategic advantage see e.g. Greenberg (2000), Binmore (2009), Bade (2011), Riedel and Sass (2014), Di Tillio et al (2017), de Castro and Yannelis (2018) among others. However, in order to exploit the strategic advantage of ambiguity players must be able to make ambiguous strategies. By making ambiguous strategies we mean a procedure which draws an element from the set of priors in a way that it does not give rise to any distribution over the priors. A player commits herself to the procedure, that is, whatever prior the procedure chooses she applies the prior to play her action. Therefore, the procedure is a generalization of randomization which can be used to make mixed strategies.

Particularly, we have in mind a situation when a tax authority (or an insurance company) introduces some uncertainty in tax audit (fraud-detection)[1]. It is common that a tax authority introduces some randomness in tax audit, e.g. a taxpayer faces tax audit with 5% chance, in order to reduce tax evasion. More recently a quite rich literature about the impact of ambiguity type uncertainty in tax audit on taxpayers' return has come out, see e.g. Snow and Warren Jr. (2005); Snow (2011); Lang and Wambach (2013); Lawsky (2013) among others.

However, in order to introduce ambiguity type uncertainty into the legal system (e.g. into the practice of Internal Revenue Service) we have to define it objectively. If a tax authority introduces ambiguity in tax audit, then this ambiguity must be well-defined and transparent, like randomization is well-defined and transparent in the sense the authority commits itself to follow a randomization device. Therefore, we need an ambiguity generator, a device which can generate objective ambiguity. In this project we have introduced such a device.

It is dominant in the literature of ambiguity that the preferences of the decision maker is represented by a non-additive probability see e.g. Schmeidler (1989); Gilboa and Schmeidler (1989); Ghirardato and Marinacci (2002);

[1]"Each year, the government faces a massive shortfall in tax collections: the annual difference between the amount taxpayers owe the government and the amount the government actually receives is nearly $400 billion dollars." by Lawsky (2013).

Gilboa (2006); Marinacci and Montucchio (2006); Maccheroni et al (2006); Gilboa (2009); Ghirardato and Siniscalchi (2010); Cerreia-Vioglio et al (2011); Gilboa and Marinacci (2016) among others. Usually non-additive probability and a concept of integral together are the mathematical ingredients of the representation results, e.g. in Schmeidler (1989) normalized capacities are the non-additive probabilities and the Choquet integral (Choquet, 1953) is the applied integral; in Gilboa and Schmeidler (1989) the non-additive probabilities are normalized exact TU games – Gilboa and Schmeidler consider convex, compact sets of probability distributions as "non-additive probabilities", however, a convex, compact set of probability distributions and (the core of) a normalized exact TU game are equivalent mathematically – and the "integral" is the maximin expected payoff.

In the case of risk, independently from that whether the applied approach is objective or subjective the mathematical representation is a probability distribution. However, in the case of ambiguity even in the subjective approach we see different mathematical representations of ambiguity, e.g. in Schmeidler (1989) normalized capacities are applied while in Gilboa and Schmeidler (1989) normalized exact TU games are used. Moreover, in Nguyen (1978); Castaldo et al (2004) the non-additive probability is a grounded, normalized, totally monotone, σ-continuous set function, therefore, even the applied notion of non-additive probability is not uniform in the literature.

In this project first we have argued for that belief measures (grounded, normed, totally monotone set functions) as non-additive probabilities are appropriate for modeling objective ambiguity. We have showed that both the random set approach – where the ambiguity type uncertainty is described by a random set – and the inner measure approach – where the ambiguity type uncertainty is described by an inner measure of a probability charge – are equivalent with applying belief measures. Particularly, we strongly believe that the inner measure approach is very strong intuitively, therefore, we have concluded as the appropriate model of objective ambiguity is the notion of belief measure.

Furthermore, we have introduced a mathematical construction (the skeleton of a device) which can generate objective ambiguity. The construction is based on the inner measure approach. Take a probability distribution (probability charge space) and extend this distribution onto all the subsets. The extension is typically not unique, and the class of extensions are given by the so called inner measure (which is a belief measure). Then we apply Stecher et al (2011)'s method to pick up an extension in a way that it does not lead to any distribution on the possible extensions. In other words, by our result and Stecher et al (2011)'s method we can draw a prior from the class of priors given by a belief function, and the procedure does not lead to any probability distribution on the priors, hence nobody can assign objective probability to the priors. We think objective ambiguity is the case when nobody can assign objective probability to the priors, and the case when nobody can assign objective probability to the priors is objective ambiguity. Naturally, a decision maker can assign subjective probability to the priors, but this probability distribution is based on the decision maker's preferences, hence it is not objective.

We have also introduced a mathematically precise notion of ambiguous strategy, which is a generalization of mixed strategy, and can be applied in strategic situations, in game theory. As in the case of mixed action in the case of ambiguous strategy when a player evaluates an strategy her attitude towards ambiguity is important. Objective ambiguity is not about attitudes, however, all the Choquet integral (Choquet, 1953) applied by Schmeidler (1989), the maximin expected payoff applied by Gilboa and Schmeidler (1989) and the concave integral applied by Lehrer (2009, 2012) lead to the very same evaluation in our model (in the case of belief functions), hence at least one evaluation method of ambiguous actions are already at hand.

Up to our knowledge the literature of applying objective ambiguity in strategic situations is not too wide. Binmore (2009) considers the strategic importance of ambiguity, and he introduces muddling boxes as source of objective ambiguity. However, he does not construct muddling boxes[2], he takes them given. Riedel and Sass (2014) consider games where the players can use a device to generate objective ambiguity in order to use ambiguity for strategic proposes. However, Riedel and Sass do not specify the details of the device, they assume that the proposed device is given. Di Tillio et al (2017) apply ambiguous strategies based on objective ambiguity in mechanism design problems. However, Di Tillio et al do not give any method to make ambiguous strategies, they take these strategies given.

Battigalli et al (2015) use ambiguity in strategic situations (games) but the source of ambiguity in their model is incentive compatibility, therefore, ambiguity is subjective in Battigalli et al (2015). In Epstein and Schneider (2007)'s model a person changes the priors, which can give rise to speculations about the person's taste etc., particularly in an interactive setting, in a game. Therefore, we think the ambiguity by Epstein and Schneider (2007)'s model is not objective. Ambiguity is not objective in Greenberg (2000) and Bade (2011) either. de Castro and Yannelis (2018) consider maximin preferences, hence they work with subjective ambiguity in their model.

Stecher et al (2011) is the closest to our model in its goal. Stecher et al (2011) introduce a method to simulate ambiguous outcomes by applying com-

[2]Following Binmore's terminology, in this paper we construct muddling boxes.

position of Cauchy random variables. Their method is suitable to simulate ambiguity in experiments, but it is less suitable for generating ambiguity in strategic situations. We use Stecher et al's method to get (real) numbers in a way, even the distribution of the numbers is not known. By these numbers we can draw an element from the set of priors given by a belief function (see above), hence we can make objective ambiguity.

In our research plan we set the following three main topics to consider:

1. Considering the Nash equilibrium (and the likes e.g. correlated equilibrium (Aumann, 1974, 1987)) and the concept of rationalizability – existence, uniqueness, etc. – when the players' beliefs are described by non-additive probabilities (ambiguity is present). These problems are a bit technical.

2. There are related results in the literature (see e.g. Riedel and Sass (2011)) but these papers do not give convincing interpretations of non-additive beliefs in the game theory setting, that is, of the non-additive mixed strategies. What does a non-additive mixed strategy mean? How can a mix be non-additive? Up to our knowledge these interpretation problems are widely open. Our goal is to give a convincing interpretation of non-additive mixed strategies by considering the interpretations of ordinary additive mixed strategies (see Harsányi (1973); Aumann and Brandenburger (1995) among others).

3. By considering the concepts of Nash equilibrium and rationalizability we can understand their relations in case of ambiguity.

The first "result" of this project was that we learned that point 2. is far most important problem than the other two. This is because without any method of making ambiguous strategies in hand, the strategic application of ambiguity is in question, hence points 1. and 3. are off till point 2. is not addressed properly.

Moreover, even the approaches of Harsányi (1973) and Aumann and Brandenburger (1995) are not adequate, since those are based on subjective risk, meaning, the mixings in these models are based on the informedness of the decision maker/player, those are not produced by a device. Again, we need objective ambiguity, hence we could not start from subjective risk.

As we have already mentioned above, after some detours, we have managed to solve this highly non-trivial problem, and we have produced a construction (skeleton of a device) which can generate objective ambiguity, hence it can be used to make ambiguous strategies. Practically, we have spent the all project to address point 2., since it turned out this problem is most important one and highly non-trivial.

I moved this project twice due to a change of my job and once the host institution suspended the project. I did not have access to my project from 28.04.2016 to March 2017 and from 1 September 2018 to January 2020. From March 2020, the pandemic made research difficult (no travelling).

Overall, with the extensions, I was able to work on the project smoothly for a maximum of 23 months, and for another 11 months with the pandemic. So, the project lasted for a little over three years, of which it was not possible to welcome visiting researchers and travel (for a conferences, short term visits) in one year.

In this project, I undertook one article a year, one of which is in Hungarian and the others in English. As the project was shortened to three years instead of four, the commitment is three articles, of which at most one is in Hungarian.

Results:

1. The main result: We have provided a method to generate objective ambiguity, by which one can make ambiguous strategies. This result is in the manuscript: "How to make ambiguous strategies" (https://d85677ad-a-62cb3a1a-s-sites.googlegroups.com/site/miklospinter/home/home/papers/StrategicAmbiguity5.pdf). This paper is in second round at Journal of Economic Theory.

2. The problem of ambiguity might be related to quantum information theory (see e.g. Aerts et al (2014) among others). We have looked at this direction in this project, the main outcome of this detour: Koniorczyk, M; Bodor, A; Pinter, M: Ex ante versus ex post equilibria in classical Bayesian games with a nonlocal resource, PHYSICAL REVIEW A 101: 6 Paper: 062115, 5 p. (2020).

3. It is well-known that ambiguity is very related to cooperative game theory in the applied mathematics. We have looked at this direction in this project, the main outcome of this detour: A cardinal convex game with empty core, MATHEMATICAL SOCIAL SCIENCES 83 pp. 9-10. , 2 p. (2016).

4. As we promised, we wrote an appetizer paper about ambiguity in Hungarian: Ismerkedés a pontatlan valószínűség fogalmával SZIGMA 51 : 4 pp. 401-413. , 13 p. (2020).

5. During the period of the project, we gave three invited lectures on the topic:

- ,,Objective ambiguity," 7th International Conference on Mathematics and Informatics, Marosvásárhely, Románia, 2019, (plenary speaker),
- ,,Objective ambiguity," Economics Seminar of the Center for Mathematical Economics at the University of Bielefeld, Németország, 2019,
- ,,Charges or measures as beliefs," Maastricht University, Hollandia, 2018,

6. We presented our results on three international conferences (one online) and on one seminar (CIAS seminar 2020 online).

   - ,,Objective ambiguity," 30th Game Theory Festival, Stony Brook, USA, 2019,
   - ,,Objective ambiguity," RUD 2019, Paris, France, 2019,
   - ,,How to generate objective ambiguity ," Annual Financial Market Liquidity Conference 2020, online,

7. We have organized three workshops in very related topics:

   - Quantum Decision Theory Workshop 2018, Pécs, Hungary,
   - AXGT 2017, 2017, Pécs, Hungary,
   - Advances in Stochastic Games, 2019, Budapest, Hungary.

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