Packing and extremal problems

Final report of the project

In this project we mainly studied embedding and packing problems in graph and hypergraph theory and problems in combinatorial discrepancy. Below we give the details of the work done during the project.

The proposed research questions were as follows:

1. Packing of trees without the Regularity lemma.

2. Random perturbation of graphs and embedding problems.

3. Embedding and packing problems for r-regular graphs with vanishing density.

4. Investigations in the discrepancy of graphs.

An important aspect was to develop methods that avoid the use of the Regularity of Szemerédi [18] whenever it is possible, as this very powerful tool requires enormously large graphs.

We give an overview of our results in the listed order of the proposed questions. At the end we give details of our other results in combinatorics that were supported by the project.

Béla Csaba introduced a graph decomposition method in "A new graph decomposition method for bipartite graphs" [7]. In that paper the edge set of a sufficiently dense graph is decomposed into lower-regular pairs, which are special bipartite pseudorandom graphs. One of the parts of the pseudorandom subgraphs of the decomposition were vertex disjoint, in the other part they may had non-empty intersections. This method was used for finding an almost decomposition of the edge set of a host graph by large trees. This result was published in the proceedings of MATCOS-19.

In the paper "Regular decomposition of the edge set of graphs with applications" [8] Csaba used the Komlós graph functional method iteratively for decomposing the edge set of a graph into super-regular pairs such that these pseudorandom subgraphs are edge disjoint, although their vertex sets may intersect. The decomposition is meaningful whenever the graph has $\Omega(n^2 \log\log n/\sqrt{\log n})$ edges, where n denotes the number of vertices in it. Hence, it may work for graphs of practical size.

This decomposition was also applied for almost tree packings (when $o(n^2)$ edges may be left uncovered) of graphs even in case the trees have (relatively small) linear size and (relatively small) linear maximum degree, and for almost packings by a set of bounded degree bipartite graphs that may have (relatively small) linear size. The only requirement for the host graph that it has $\Omega(n^2 \log\log n/\sqrt{\log n})$ edges in case of tree packings, and $\Omega(n^2 \log\log n/\sqrt[5D]{\log n})$ in case the graphs to be packed have maximum degree D. So these results are closely related to the celebrated Gyárfás-Lehel conjecture and to the Oberwolfach problem. While these theorems are on almost decompositions, in some aspect they are stronger than the previously mentioned problems, as the host graph does not have to be a complete graph or a complete bipartite graph.

In another application of this decomposition the following question was considered: Given a 3-partite graph on n vertices with a "few" cycles of length 5, how many edges have to be removed from it to make it triangle-free? This question is a version of the famous removal lemma, first proved by Ruzsa and Szemerédi [17], then improved by Fox [12]. The above type of question, a conditional removal lemma, was asked by Conlon et al. [6] for C_4-free graphs. Csaba proved, that by deleting at most ϵn^2 edges, the graph can be made triangle-free, if the number of C_5s in it is at most $exp(-c/\epsilon^2)n^5$, where $c > 0$ is a constant. This result is in contrast with the unconditional removal lemma, in which one needs a tower-type bound.

An algorithmic version was also proved: there exists a randomized algorithm, that runs in polynomial time and finds a decomposition of the edge set into large pseudorandom subgraphs. In this decomposition the density bound for the graph is larger, but a vanishing density is still possible. This paper is submitted for publication.

In "Bundle decompositions of graphs with algorithmic applications" [9] the algorithmic decomposition theorem is used for three NP-complete optimization problems: 2-VC, 2-EC and TSP(1, 2). In the 2-VC problem we want to find the smallest 2-connected spanning subgraph of a given graph, while in the 2-EC problem we look for the smallest 2-edge-connected spanning subgraph. In the TSP(1, 2) problem we have a complete graph with edge weights 1 or 2, and need a Hamilton cycle with the smallest total edge weight. In this question the input graph is the subgraph spanned by edges of weight 1. Csaba proved that if the input graph in any of the above three questions has n vertices and is r-regular for some $r = \Omega(n \log\log n/\sqrt[k]{\log n})$, then there is a randomized polynomial time algorithm that can approximate the optimal solution with arbitrary precision. This paper is submitted for publication.

Béla Csaba and András Pluhár, with József Balogh and Yifan Jing, considered a problem in combinatorial discrepancy theory in [2]. We are given a graph G on n vertices and a function f that assignes -1 and $+1$ to the edges of G. For a subgraph H of G the discrepancy of H in G is the sum of the f-values of the edges of H, minimized for f. The question is: how large discrepancy cannot be avoided when H is the set of all Hamilton cycles of G or the set of its spanning trees, etc? Here one also imposes conditions of G, for example, G has large minimum degree, or G is the $k \times k$ grid. For example, they proved that if the minimum degree of G is at least $(3/4 + c)n$ for a $c > 0$, then linear discrepancy is unavoidable, but there are graphs with minimum degree precisely $3n/4$ with zero discrepancy. This paper appeared in the Electronic Journal of Combinatorics.

Béla Csaba and András Pluhár with József Balogh and Andrew Treglown in [3] studied a discrepancy version of the Hajnal-Szemerédi theorem. Define a perfect r-clique-tiling in a graph G to be the collection of vertex-disjoint copies of the r-clique in G covering every vertex of G. The famous Hajnal–Szemerédi theorem determines the minimum degree threshold for forcing a perfect r-clique-tiling in a graph G. The following discrepancy version of the Hajnal-Szemerédi theorem is considered in the paper. One assigns the edges of a graph G labels from $\{-1, 1\}$, and one seeks substructures F of G that have 'high' discrepancy (i.e. the sum of the labels of the edges in F is far from 0), where the F substructures are the perfect r-clique-tilings. In this paper they determine the minimum degree threshold for a graph to contain a perfect r-clique-tiling of high discrepancy. Their paper appeared in the journal Combinatorics, Probability and Computing.

Béla Csaba and Judit Nagy-György worked on a graph embedding problem in [5]. They published the following result: if the Ore-degree of an n-vertex graph is at most 5, then it can be embedded in any n vertex graph having minimum degree at least 2n/3. These degree bounds are sharp. The paper appeared in the SIAM J. Disc. Math.

Csaba with his doctoral student Bálint Vásárhelyi studied the following embedding question in [10]. Assume that we are given two graphic sequences, p1 and p2. What conditions for p1 and p2 guarantee that there exists a simple graph H realizing p2 such that H is the subgraph of any simple graph G that realizes p1. This result appeared in the journal Informatica.

Csaba and Vásárhelyi published another paper in extremal graph theory [11]. They constructed a class of bounded degree bipartite graphs with a small separator and large bandwidth. This proves that while separability and bandwidth are sublinearly equivalent, there is no linear equivalence of the two notions. Moreover, they also proved that graphs from this class are spanning subgraphs of host graphs with minimum degree just slightly larger than $n/2$. This paper appeared in the journal Graphs and Combinatorics.

Péter Hajnal, with Endre Szemerédi, worked on the semi-random method [16]. The semi-random method was introduced in the early eighties, in its first form it was used to give lower bounds for the size of the largest independent set in so called uncrowded hypergraphs. The first geometrical application was a major achievement in the history of Heilbronn's triangle problem. It proved that the original conjecture of Heilbronn was false. In this paper Hajnal and Szemerédi gave two further geometrical applications. First, they improved on Payne and Wood's upper bounds on a Ramsey-type parameter which was introduced by Gowers. Secondly, they improved Schmidt's bound on Heilbronn's quadrangle problem. Their paper appeared in the János Bolyai Society Lecture Notes.

Péter Hajnal, with Zhihao Liu and György Turán, studied nearest neighbor representations of Boolean functions [15]. A nearest neighbor representation of a Boolean function is a set of positive and negative prototypes in the n-dimensional space such that the function has value 1 on an input if and only if the closest prototype is positive. For k-nearest neighbor representation the majority classification of the k closest prototypes was considered. The nearest neighbor complexity of a Boolean function is the minimal number of prototypes needed to represent the function. They gave several bounds for this measure, together with separations between the cases when prototypes can be real or are required to be Boolean. The complexity of parity was determined exactly. They gave an exponential lower bound for mod 2 inner product, and a linear lower bound for its k-nearest neighbor complexity. The results were proven using connections to other models. This paper was accepted for publication in Information and Computation.

Péter Hajnal, with Simao Herdade and Endre Szemerédi, considered the Pósa-Seymour conjecture, a very hard embedding problem in graph theory [14]. Earlier this conjecture was proved by Komlós, Sárközy and Szemerédi with the help of the Regularity Lemma - Blow-up Lemma method, and therefore was known to hold only for huge graphs. In their work they present another proof that avoids the use of the Regularity Lemma and thus resulting in a much smaller threshold for the size of the graph in question. Equally important is that they prove a stability result about this conjecture. The main ingredient of their proof is a new kind of connecting lemma. The paper on the result is in preparation.

Judit Nagy-György with Debarun Ghosh, Ervin Győri, Addisu Paulos, Chuanqi Xiao and Oscar Zamora considered book free 3-uniform hypergraphs [13]. One of the first problems in extremal graph theory was the maximum number of edges that a triangle-free graph can have. This was solved by Mantel, which built the foundations of what we know as extremal graph theory. The natural progression was to ask the maximum number of edges in a k-book free graph. A k-book, denoted by B_k, is k triangles sharing a common edge. The hypergraph analogue of the maximum number of hyperedges in a triangle-free hypergraph was solved by Ervin Győri in the early 2000s. In a hypergraph, k-book denotes k Berge triangles sharing a common edge. Let $ex_3(n, F)$ denote the maximum number of hyperedges in a Berge-F-free hypergraph on n vertices. In this paper they proved that $ex_3(n, Bk) = (1 + o(1))n^2/8$. The paper was submitted for publication.

Gábor V. Nagy with Beáta Bényi considered fillings of Ferrers shapes. They showed that Γ-free fillings and lonesum fillings of Ferrers shapes are equinumerous by applying a previously defined bijection on matrices for this more general case and by constructing a new bijection between Callan sequences and Dumont-like permutations. As an application, they gave a new combinatorial interpretation of Genocchi numbers in terms of Callan sequences. Furthermore, they recovered some of Hetyei's results on alternating acyclic tournaments. Finally, they presented an interesting result in the case of certain other special shapes. Their paper with the title of Lonesum and Γ-free 0-1 fillings of Ferrers shapes appeared in the European Journal of Combinatorics.

Gábor V. Nagy with Horst Alzer considered identities involving central binomial coefficients and Catalan numbers. Their main results is a sophisticated identity of this kind. They proved it combinatorially, and also presented a short algebraic proof. Among others they also considered series representations for the reciprocal of π and related constants. Their paper titled Some identities involving central binomial coefficients and Catalan numbers appeared in the journal INTEGERS.

Research trips, conference visits, and therefore a wide dissemination of our results were also part of the project. Unfortunately, due to the pandemy, except one conference (MATCOS-19 in October 2019, Koper, Slovenia) all the others we planned to visit were postponed or cancelled.

References

[1] H. Alzer, G. V. Nagy, Some Identities Involving Central Binomial Coefficients and Catalan Numbers, INTEGERS 20, Paper #A59. (2020) URL: http://math.colgate.edu/ integers/u59/u59.pdf

[2] J. Balogh, B. Csaba, Y. Jing and A. Pluhár, On the discrepancies of graphs, Electron. J. Combin. **27** (2020) P2.12., see also at https://arxiv.org/pdf/2002.11793.pdf

[3] J. Balogh, B. Csaba, A. Pluhár, A. Treglown, A discrepancy version of the Hajnal–Szemerédi theorem COMBINATORICS PROBABILITY & COMPUTING **30** (2021) 444-459, see also at https://arxiv.org/pdf/2002.12594.pdf

[4] B. Bényi, G. V. Nagy, Lonesum and Γ-free 0-1 fillings of Ferrers shapes, EUROPEAN JOURNAL OF COMBINATORICS 89 (2020) DOI: https://doi.org/10.1016/j.ejc.2020.103180 arXiv: https://arxiv.org/abs/1911.10342

[5] B. Csaba, J. Nagy-György, Embedding Graphs Having Ore-Degree at Most Five SIAM JOURNAL ON DISCRETE MATHEMATICS **33** (2019) 474-508. see also at https://arxiv.org/pdf/1707.07216.pdf

[6] D. Conlon, J. Fox, B. Sudakov, Y. Zhao, The regularity method for graphs with few 4-cycles, arXiv preprint arXiv:2004.10180.

[7] B. Csaba, A new graph decomposition method for bipartite graphs, Proceedings of MATCOS 2019, 11-14., see also at https://arxiv.org/pdf/2109.12429.pdf

[8] B. Csaba, Regular decomposition of the edge set of graphs with applications, https://arxiv.org/pdf/2109.12394.pdf, submitted for publication.

[9] B. Csaba, Bundle decompositions of graphs with algorithmic applications, https://www.math.u-szeged.hu/ csaba/cikkek/NyalabRegParticio.pdf, submitted for publication.

[10] B. Csaba, B. Vásárhelyi, On embedding degree sequences INFORMATICA-JOURNAL OF COMPUTING AND INFORMATICS **43** (2019) 23-31. see also at https://arxiv.org/pdf/1904.11778.pdf

[11] B. Csaba, B. Vásárhelyi, On the Relation of Separability, Bandwidth and Embedding GRAPHS AND COMBINATORICS **35** (2019) 1541-1553. see also at https://arxiv.org/pdf/1904.11778.pdf

[12] J. Fox, A new proof of the graph removal lemma, Annals of Mathematics (2) 174 (2011), 561-579.

[13] D. Ghosh, E. Győri, J. Nagy-György, A. Paulos, C. Xiao, O. Zamora, Book free 3-uniform hypergraphs, see at http://www.math.u-szeged.hu/ ngyj/link.html

[14] P. Hajnal, S. Herdade, E. Szemerédi, Proof of the Pósa-Seymour conjecture, in preparation.

[15] P. Hajnal, Z. Liu, G. Turán, Nearest neighbor representations of Boolean functions, to appear in Information and Computation, see also at https://arxiv.org/pdf/2004.01741.pdf

[16] P. Hajnal, E. Szemerédi, Two geometrical applications of the semi-random method, New trends in intuitive geometry, 189-199, Bolyai Soc. Math. Stud., 27, János Bolyai Math. Soc., Budapest, 2018., http://www.math.u-szeged.hu/~hajnal/research/papers/2geo_semirandom.pdf

[17] I. Ruzsa, E. Szemerédi, Triple systems with no six points carrying three triangles, Combinatorics (Keszthely, 1976), 18 (1978), Vol. II., 939-945. North-Holland, Amsterdam-New York.

[18] E. Szemerédi, Regular partitions of graphs, Colloques Internationaux C.N.R.S. No 260 - Problémes Combinatoires et Théorie des Graphes, Orsay (1976), 399-401.