Final report

on the project SNN 129364

In the framework of the project SNN 129364, we achieved many new scientific results. The list of papers acknowledging support from this project includes 124 items. In SCI journals with impact factor, 80 papers have been published, plus 7 are accepted for publication; and 25 others are under review, also at high-level international journals. Further 12 manuscripts are published in refereed non-SCI journals and conference proceedings.

The joint papers published with the Slovenian research group are mostly in the area of graph domination theory and in the study of structures obtained via graph product operations. Tendency indicates that some of these papers will be among the highly cited works of the authors. Collaboration between the two research groups has been strengthened with joint participation in conferences and workshops. Moreover, Balázs Patkós and Zsolt Tuza were invited speakers at Slovenian conferences, and Bostjan Bresar gave an online talk at the combinatorics seminar of the Rényi Institute during the pandemy.

During the duration of the project, many aspects of Turán-type problems and related areas have been investigated. The most important results are summarized below.

1. TURÁN-TYPE PROBLEMS

The Turán number of a graph G is the maximum number of edges in an n-vertex G-free graph. It is a central parameter in extremal graph theory, and the problem has many generalizations and variants.

1.1. Singular and regular Turán number of a graph G.

A subgraph G of H is called singular if its vertices either have the same degree in H or have pairwise distinct degrees in H. The largest number of edges of an n-vertex graph that does not contain a singular copy of G is the singular Turán number of G. In the first paper, we established strong general asymptotic results. We later determined the singular Turán number of the triangle for every n of the form 4k and 4k+1, and further the singular Turán number of the r-clique for values of n divisible by r-1.

We also introduced another variant of the classical Turán number, the regular Turán number of a graph G that is the maximum number of edges that a regular graph on n vertices can have without containing a copy of G. We determined the regular Turán number of complete graphs and trees, for every n sufficiently large with respect to G.

1.2. Generalized Turán problem

In the generalized Turán problem, we are given two graphs H and F, and wish to determine the largest number of copies of H in an F-free n-vertex graph. After several sporadic results, this problem has attracted a lot of attention recently. It is also strongly connected to hypergraph Turán problems. Similarly to the ordinary Turán problem, only the order of magnitude or the asymptotics is known for most cases. We have studied the situation when the Turán graph gives the exact maximum, at least for large n. When a complete graph is forbidden, we obtained a class of graphs H where this is the case. Then we extended this result to the case of some other forbidden graphs. We established exact results for basic problems, e.g., where F is an odd cycle and H is an even cycle or a path. We studied the case where both H and F have at most four vertices, and for each pair of such graphs we either obtained an exact result or showed that such a result would imply a major breakthrough in other areas of extremal Combinatorics. The so-called supersaturation phenomenon for these problems was also investigated. We proved that if an n-vertex graph G contains more copies of H than the above bound in F-free graphs, it contains not only one but many copies of F.

1.3 Berge-Turán problem

A hypergraph is a Berge copy of a graph F if we can obtain it by replacing every edge of F with a hyperedge containing that edge. Turán problems for hypergraphs are notoriously difficult, but the well-developed graph theoretic methods can be used for Berge hypergraphs. Our main new contribution to this area is connecting it to another area that has attracted many researchers recently, the generalized Turán problems. There we consider F-free graphs, but we aim to maximize the number of copies of another subgraph H rather than the number of edges. We showed that counting the r-cliques in F-free graphs and counting the hyperedges in r-uniform Berge-F-free hypergraphs are closely related. We obtained new bounds for the Turán number of Berge trees, cliques, cycles, and other graphs by further exploring this connection.

Another result settles the general case of the weighted non-uniform version of this problem. For almost all weight functions, we have determined how large the order of magnitude of the total weight of a Berge-F-free family can be, provided that the hyperedges are large enough. We have also studied several variants and generalizations. We examined the case where we start with a hypergraph instead of a base graph F. We studied a t-wise version, where t distinct hyperedges replace every edge.

We also studied Berge-Turán problems in which the host hypergraph is linear.

1.4 Connected Turán problem

We also investigated the version of the problem where the maximum number of edges has to be determined in a connected graph of order n that does not contain a given graph H as a subgraph. This connected Turán number differs from the ordinary one when H is a forest. We also proved that the connected Turán number can be much smaller than what the Erdős-Sós conjecture states for the ordinary Turán number.

1.5 Turán problem for q-graphs and related colorings

We studied the intersection properties of vectors with entries from p{0,1,...,q}, generalizing intersection problems corresponding to the q=1 case and adapted this context to extremal graph theory. The incidence matrix of a graph is a 0-1 matrix, with the rows corresponding to the vertices and the columns corresponding to the edges, and the entry is 1 if the edge of the column contains the vertex of the row. The adjacency matrix of a 'q-graph' is a matrix with entries from {0,1,...,q} such that all columns contain exactly two non-zero entries. We defined the property of such a q-graph containing an s-sum copy of an ordinary graph and the Turán problem that asks for the maximum number ex(n,q,s,H) of edges (columns) in an n-vertex q-graph that does not contain any s-sum copy of a forbidden ordinary graph H. The q=1, s=2 case corresponds to the original Turán problem. We obtained many results in the special case s=q+1.

One of our tools was a newly introduced coloring concept of ordinary graphs. Namely a quite different but equivalent formulation of the definition requires a vertex coloring where each color class induces a subgraph such that every component contains at most one cycle. We studied the basic properties of this so-called robust coloring number and established connections with well-known graph parameters such as maximum degree, degeneracy, clique number, and covering number. The robust chromatic number over different graph classes (e.g., chordal graphs, split graphs, complete multipartite graphs, Kneser graphs) was also investigated thoroughly.

1.6 Stability problems

We proved the following stability result: edge-maximal F-free graphs with an almost maximum number of edges always contain an almost spanning complete r-partite graph if the chromatic number of F is r+1. Concerning Berge-F-free graphs, we determined the exact value of the 3-uniform Turán number of Berge copies of book graphs. The Berge uniformity threshold of a graph F is the smallest integer r such that r-uniform Berge-F-free hypergraphs on n vertices have o(n^2) edges. We obtained new bounds for this threshold and determined its value for some classes of graphs.

In another setting, for a given pair H and F of graphs, the generalized Turán problem asks to determine the maximum number of copies of H that an n-vertex F-free graph G may contain. We considered this problem and proved theorems for the cases where H=K_2,t, or both H and F are complete bipartite graphs, or H or F is a double-star. We also studied a special case where G may contain F, but we only count copies of H the vertex set of which does not contain and is not contained in the vertex set of any copies of F.

2 RELATED TOPICS FROM EXTREMAL COMBINATORICS

2.1 Saturation problems

Saturation problems are natural counterparts of extremal problems. Given a property of some combinatorial structures, one is interested in the minimum possible size of a structure that possesses this property, but adding any new element (an edge, a hyperedge, a set) would ruin this property. In the graph setting, we considered three variants of regular H-saturated graphs G:
- G is H-free and the addition of any edge to G would create a copy of H;
- G is H-free and any regular supergraph of G contains a copy of H;
- G is not necessarily H-free, but any new edge creates a new copy of H.
In the set system setting, we obtained two similar and somewhat surprising results.

The Vapnik-Chervonenkis dimension is a well-studied notion applied in different areas of mathematics. We proved that for any integer $d>1$, there exists a constant C depending only on d such that for any n there exists a family F of at most C subsets of $[n]=\{1,2,...,n\}$ with VC-dimension d such that adding any new subset of [n] to F increases the VC-dimension. Similarly, concerning the saturation version of the forbidden subposet problem, we obtained that for any finite poset P there exists a constant C depending only on P such that for any n there exists a family of at most C subsets of [n] that is weak P-free, but adding any new subset of [n] creates a weak copy of P.

The following question was also studied: for which graphs F does there exist for every large enough n an n-vertex regular F-saturated graph, and if there are such graphs for F, then what is the minimum number of edges they may contain as a function of n? We examined some related relaxed problems as well, and obtained several constructions for the triangle and other complete graphs, based on sum-free sets known from number theory. Moreover, we proved for an infinite class F of graphs (including all complete multipartite graphs) that the saturation number of r-uniform Berge-F-free hypergraphs is linear.

2.2 Forbidden subposets

Concerning the original extremal version of forbidden subposet problems, we introduced three strengthenings of the main conjecture of the area and proved them in some special cases:
- what is the minimum number of weak copies of P in a set system containing E extra sets compared to the extremal number when E is proportionate to the size of the middle level of the Boolean lattice;
- what is the maximum size of a P-free subfamily of the p-randomized Boolean lattice P(n,p), i.e. every subset F of [n] belongs to P(n,p) with probability p, independently of all the other subsets;
- what is the number of all P-free families.
To each of these questions a conjecture was stated, which would imply the widely believed conjecture on the size of the largest P-free family.

Improving on earlier results, we determined the order of magnitude of the maximum number of k-chains in P-free families for posets P of height 2. In some special cases as the butterfly poset and some path-like posets we managed to obtain exact results. We extended our research on the forbidden subposet problem to host posets other than the Boolean poset. For the d-dimensional grid, we established a connection to the set family case, and solved some of the most natural instances of the problem. The saturation version was also examined and compared to the corresponding forbidden 0-1 matrix problem.

2.3 Ordered Ramsey numbers

There were no previous investigations of Ramsey-type questions for vertex-ordered hypergraphs. In this direction we proved fairly good bounds on ordered Ramsey numbers of multipartite graphs.

2.4 Majority problem

Majority problem is a combinatorial search problem, also of an extremal type, where one has to find a majority element among two colored items. (A majority element is one from the color class whose size is strictly larger than the others.) We investigated different models of the majority problem in the non-adaptive case.

3 FURTHER RESULTS

3.1 Decompositions of hypergraphs

Decomposig the edge set of complete uniform hypergraphs is a major issue within design theory. On the other hand, except for the 2-uniform case (graphs), very little is known about decompositions where a 1-factor (parallel class) is omitted from the family of k-element sets. For such structures in the 3-uniform case it was proved already 60 years ago that no decompositions can exist into cycles of length 4. This negative result had no analogue since then. After six decades our positive new results are the first to prove that complete 3-uniform hypergraphs minus a 1-factor admit decompositions into cycles of lengths 6, and also into cycles of length 9, whenever the necessary divisibility conditions are satisfied. We also built many new constructions and recursive rules for decompositions of the complete 3-uniform hypergraph into 5-cycles.

3.2 Intersecting set families

We considered non-trivially intersecting set families with their profile vectors and determined the extreme points of their profile polytopes. We also introduced a generalization of set intersection via the characteristic vectors of sets. Two vectors of length n both with entries from {0,1,...,q} s-sum t-intersect if there exist at least t coordinates where the entries sum up to at least s. So ordinary set intersection corresponds to the case q=1, s=2. For any given q,s,t,k and n large enough, we determined the maximum number of vectors of length n that a k-uniform s-sum t-intersecting family of vectors can contain. We also addressed the vector analog of Bollobás's intersecting set-pair system problem and obtained lower and upper bounds that generalize known results in the set family setting.

3.3 Vertex sets in general position

In geometry, a point set is said to be in general position if no three points of the set lie on the same line. There are several ways to reformulate not being collinear, one of which is having strict inequality in the triangle inequality for every triple of elementsv in the set. This definition generalizes to graphs and one says that a subset U of the vertices of a graph G is in general position if no three vertices of U satisfy the triangle inequality with equality with respect to the graph distance in G. For particular graphs (hypercubes, toroidal grids) there had existed some sporadic results on the maximum possible size of a vertex set in general position, but systematic investigation of this quantity has only started recently. Our contribution to the area includes determining the exact threshold n(k) such that the unique largest vertex set of the Kneser graph K(n,k) in general position corresponds to the star if and only if n is at least n(k). In another manuscript, we deduced the general position number of Cartesian products of paths and cycles, and gave probabilistic lower bounds on the general position number of Hamming graphs and of Cartesian graph powers.

3.4 Domination games

The domination game, a competitive optimization version of graph domination, was introduced ten years ago and got considerable attention. We extended this definition in different directions:

We studied the domination game played on hypergraphs. An asymptotically tight upper bound, in terms of k and the order n, was established on the game domination number of k-uniform hypergraphs. We proved that the game domination number of an isolate-free 3-uniform hypergraph cannot be greater than 5n/9.

Studying another direction, we introduced the fractional version of the domination game, where the condition of fractional domination rules the moves. Among several fundamental properties of this new game, we established the fractional version of the so-called Continuation Principle. Lower and upper bounds on the fractional game domination number of paths and cycles were determined. These estimates are tight, apart from a small additive constant.

In a third paper, the Z-, L-, and LL-domination games were introduced, which are natural companions of the standard and total domination games. We got the following main results: versions of the Continuation Principle for the new games, hierarchy of the five domination games, and values of the three new invariants for paths, up to a small additive constant independent of the length of a path. On the non-game version, extremal and structural results have been achieved concerning Boolean lattice graphs and k-trees.

3.5 Combinatorial optimization

Related to this area, we designed and analyzed various algorithms. We dealt with a production planning problem, more exactly we investigated a special version of scheduling unrelated machines with makespan minimization. We also scheduled coupled tasks on a single machine. Here any job consists of two tasks, and a given exact delay time must elapse between the two tasks. Given are many such tasks with only two different delay times, and unit execution times for the task. We applied the appropriate version of the First

Fit Decreasing algorithm and provided almost tight bounds for its worst-case performance. Also, parameter optimization of a reinforcement learning algorithm to increase its efficiency has been investigated.

For bin packing related problems, we introduced certain greedy-type pre-solve algorithms, that can optimally solve a high proportion of benchmark problems within certain types (e.g. the Schwerin-type instances). These algorithms can be considered as the first phase of a two-phase method. First we solve (optimally) a significant proportion of a set of instances, and then we need to solve only the remaining cases with some more sophisticated method. On the other hand we could find interesting and theoretically significant lower bounds for the performance of online bin packing algorithms.

We considered a generalization of the Bin Packing Problem, where the goal is not "only" to pack the items into bins, but the bins will be delivered. This option is optimized with the help of introducing a special objective function.

We also studied rectangle covering problems. In one of them we maximize the size of a square that can be covered with a given collection of small squares. In another problem a given rectangle (called board) is divided into cells, and each cell has a gain value. A collection of rectangles should be allocated within the board, so that the total covered gain is maximized. In case of both problems, we apply several kinds of algorithms, to solve the smaller and bigger (and so easier and harder) cases.