Our symmetry-related investigations can be grouped into three families.

1. SU(3)

In the year we started the present project the nuclear SU(3) symmetry had its 60th birthday. In [1] we gave a brief review of some of its important historical, present, and possible future aspects.

SU(3) is the symmetry of the space part of the nuclear wave function. The spin-isospin sector is characterized by Wigner's SU(4) symmetry [18]. Elliott's SU(3) model became very important for different reasons. (i) It gives a good description of the spectra of light nuclei. (ii) It revealed the common intersection of the shell, collective and cluster models for a single major-shell-problem. (iii) Being the prototype of the algebraic models, it showed the way to many other theoretical approaches.

SU(3) has many exciting new features, even after 60 years of hard and successful work in nuclear structure physics. Our project also serves as an illustration to its deep content and robust nature.

In a series of studies we revealed some hidden aspects of this essential symmetry and extended it in several directions, as follows.

In [2] and [7] we showed how the spontaneous symmetry breaking takes place in the Elliott model, as well as many other algebraic models. It is an interesting historical fact that the relation of the nuclear deformation and the spontaneous breaking was not discussed beforehand within the Elliott model, despite the fact that it was the first model to explain the deformation from the viewpoint of the spherical shell model. Neither in articles, nor in the textbook by Elliott and Dawber, which explains this phenomenon in different areas of physics. In [2,7] we showed the connection of the spontaneous breaking not only to the quadrupole deformation, but also to the octupole one and to the molecular configuration. Furthermore, being related to the well-known rotational symmetry and a simple Hamiltonian, the Elliott model seems to provide us with the simplest example for the spontaneous breaking, which might be relevant even for educational purposes.

In [8] we pointed out that in algebraic structure models a dual symmetry-breaking is a general feature. In particular, the larger symmetries are dynamically broken, by some symmetry-breaking interactions; while the rotational symmetry is spontaneously broken in the eigenvalue-problem of the intrinsic Hamiltonian. In particular, in the Elliott model U(3) and SU(3) are the dynamically broken larger symmetries, and SO(3) breaks spontaneously.

We have extended the historical SU(3) symmetry along different directions. For the large deformations we investigated the behavior of the quasy-dynamical U(3) symmetry in [3]. This is a generalized version of U(3), in which the eigenvectors contain linear combinations of basis sates of inequivalent irreducible representations (irreps). One can calculate them from the Nilsson-model, which contains symmetry-breaking interactions, too. It turns out that for certain deformations the U(3) symmetry becomes very well-defined and stable, characterizing the shape isomers of the nuclei. Therefore, we found a new method for the determination of the stable deformations, which is an alternative of the well-known energy-minimum calculation. It is called SCS (symmetry consistency and stability) method. This U(3) symmetry is an emergent symmetry in the sense that it comes from a calculation of the eigenvectors of non-symmetric Hamiltonian. It is very remarkable that recently the emergence of the SU(3) symmetry was found from large-scale shell model calculations, too, even with realistic interactions. These results show that the nucleus contains a few well-deformed stable shapes (and U(3) symmetries), with vibrational and rotational excitations associated to them.

The original U(3) symmetry is valid only for light nuclei. In heavier ones it breaks down due to the larger weight of the symmetry-breaking interactions, e.g. spin-orbit. Much effort has been put in finding approximate symmetries, which are valid also for heavy nuclei. The pseudo and proxy SU(3) schemes are two such approaches. The previous one is known for a long time, while the proxy scheme is a new one. In [5] we showed how the proxy-SU(3) approximation can be formulated in the spherical shell-scheme. (Previously it was known only for the asymptotic Nilsson basis.) In [6] we discussed how the algebraic quartet model (invented previously by us for light nuclei) can be extended to the heavy mass region, based on the pseudo and proxy U(3) symmetry.

The Elliott model gives a uniform moment of inertia for all the rotational bands of a nucleus. (In its original form it was applied only to a few bands.) Due to the more recent extensions the model is applied for many bands, with different deformations, therefore, a more detailed treatment of the moment of inertia is desirable. In [25] we found a simple functional form of the invariant operator of SU(3) which can give a good description of the moment of inertia of bands of different nature.

A very important extension of the Elliott model and the SU(3) symmetry is related to the incorporation of many major shells. As mentioned before the original model deals with a single-shell problem, but for many physical phenomena this reduced model space is not sufficient.

In [4] we have carried out a two-shell calculation for nuclei of mass number 32 and 36, including not only symmetry-preserving quadrupole interactions, but also symmetry-breaking (isoscalar and isovector) pairing interactions. In this way the description of the negative parity states could be reached as well.

2. MUSY

The intersection of the shell, collective and cluster models for the many major-shell-problem was found to be a new symmetry, called multiconfigurational dynamical symmetry (MUSY) [13,15,19,27]. It has an algebraic structure of $U_s \otimes U_e(3) \supset U(3)$, i.e. it is a simple extension of the historical U(3) dynamical symmetry of Elliott et. al. The subscript s refers to the valence shell in the (symplectic) shell model and in the microscopic collective model (called contracted symplectic model), and to the internal cluster structure in the cluster picture. $U_e(3)$ describes the major shell excitations. This symmetry is able to give a unified description of the spectra of different configurations (shell, quartet, various clusters) in a large range of the energy and deformation. Due to this feature it has a considerable predictive power, too. The basic idea and some fundamental features of MUSY had been invented by us previously, in the present project we worked out important details of it. Furthermore we studied how well this new symmetry is realized in Nature.

In [16] we have given the detailed mathematical background of that kind of MUSY, which is based on the invariance with respect to the transformations between different configurations. It turns out that in addition to the group structure mentioned above we need to pay attention to another one as well, which treats the particle indices explicitly. It gives the way for the construction of the invariant interactions.

In [24] we have discussed the logical structure and the possible versions of MUSY together with the supersymmetry (SUSY). These are composite symmetries of composite systems in the sense that the physical system consists of two or more components, each of them having a usual dynamical symmetry, and further symmetries connect the components. Some analogies with the SUSY and the local gauge invariance of the particle physics are also revealed.

MUSY investigations shed a new light on the long-standing discussion of shell-cluster competition or duality in atomic nuclei [13,15,19,24,27].

In the analysis of experimental data the main advantage of MUSY is that it gives a unified description of spectra scattered in excitation energy, deformation, and structural configuration. Typically it is able to predict the shape isomers, determine the possible binary reaction channels which can populate them, and describe the detailed energy spectrum from ground state up to very high energies. It covers regions, which are usually described by different models.

We have predicted the possible shape isomers of all the alpha-like (N=Z=even) nuclei in [29], determined their possible binary clusterizations (reaction channels), and gave their energy spectra. Those data which are obtained also from energy-minimum calculations show good agreement. For the superdeformed state of ^{28}Si and hyperdeformed of ^{36}Ar some experimental candidates have been proposed and further investigations are in progress.

We have obtained the detailed low-lying shell spectrum together with those of alpha clusters at exotic (^{12}C, ^{16}O) clusters in ^{20}Ne, ^{24}Mg, ^{28}Si, ^{36}Ar, and ^{44}Ti [13,14,15,16,25,27]. In ^{28}Si the high-lying ^{12}C+^{16}O spectrum turned out to be in the second (superdeformed) minimum, and MUSY gave a parameter-free prediction for the detailed spectrum, based on the description of the low-energy shell structure. The distribution of the core+alpha state is also very similar to the MUSY prediction. In ^{36}Ar the spectra of the first second and third minimum is treated in a unified way. In the ^{24}Mg MUSY calculations [26] revealed that the 0^+ resonances found recently do not support the idea of extending the Hoyle-state paradigm to the ^{12}C+^{12}C system, based on the simple cluster configuration. (But it is in line with the concept of the fragmented cluster states due to the coupling to the background quartet states.)

3. Other symmetries

3.1. SU(4)

The spin-isospin part of the nuclear wave function can be characterized by Wigner's SU(4) symmetry. It is even older than the space symmetry of SU(3). Nevertheless, many interesting questions related to it still wait for answers. We have addressed some of them both from the experimental and from the theoretical side.

One of the most direct possibility for the experimental check of this symmetry is provided by the Gamow-Teller transitions; in particular, these transition can take place only between the members of a supermultiplet [18]. In [10] we have investigated the Gamow-Teller strength distribution of the decay of ^{186}Hg to ^{186}Au by the total absorption gamma spectroscopic technique. Our results show for the very first time the possible mixed nature of the ground state and a larger prolate component.

The beta decay of the near drip-line ^{70}Kr was studied at RIKEN Nishina Center [21]. In this work we show that the beta decay strength can be nicely described using a pseudo-SU(4) symmetry schematic model, which shows the recovery of the symmetry for this N=Z+2 system.

A systematic study of the applicability of the pseudo-SU(4) symmetry for the description of the beta decay of N=Z+2 systems was performed in [30], which shows that it is applicable to A=58, it breaks for heavier masses and it is recovered for the A=70 system.

3.2. Symmetries and entanglement

Investigation of non-classical correlations between subsystems has gained importance in several research areas. The one-mode entropy is a new and useful tool to characterize quantum correlation of many body quantum systems. The entanglement properties of the two-nucleon problem has been investigated in [11] for the symmetry $SU(2) \otimes SU(2)$. In this case the $[j \otimes j]_0$ angular momentum zero states have the largest total correlation. We also studied the seniority nuclear model [20]. In this case, identical fermions occupy a single j shell. The relevant group chain is $U(2j+1) \supset Sp(2j+1) \supset SU(2) \supset U(1)$ or an alternative one using the notion of quasi-spin is the following $SU(2) \otimes SU(2) \supset SU(1) \otimes U(1)$. In both of the studied models analytic expressions are derived for the one-mode entropies. The peculiar behavior of the half-filled shells and the seniority-zero states are emphasized.

3.3. Potentials

We investigated the Bohr Hamiltonian, which describes the quadrupole collective deformation of nuclei in terms of a two-dimensional potential surface. Using our model based on the sextic oscillator potential, we studied the isotope chains of Pt and Os nuclei and identified a transition between the spherical and the so-called gamma-unstable shape phases [31].

We reviewed the developments concerning the application of the sextic oscillator in the Bohr Hamiltonian since 2004. It has been generalized along various lines both by us and other authors. Our review reports on about 30 publications containing results for about 100 nuclei [23].

We also worked on extending the range of exactly solvable potentials beyond the Natanzon potential class. We constructed polynomial solutions of the confluent Heun differential equation, from which the generalization of the Natanzon potential class can be obtained. We recovered all known solvable potentials expressed in terms of generalised Laguerre polynomials, and identified two new polynomials bearing the characteristics of semi-classical orthogonal polynomials [22].

We formulated conditions under which the members of this general potential class exhibit PT symmetry, both unbroken and broken [12]. We applied supersymmetric quantum mechanics to connect the PI subclass of Natanzon potentials with a subclass of the generalized potential class, associated with X1-type Jacobi polynomials. A wide network of SUSY transformations has been identified [32].

3.4. Celestial mechanics

Symmetries are important both in classical and in quantum physics. In the latter one they show a richer structure. Some concepts e.g. are not even apply to classical problems. The dynamical algebra was such an example for a long time. It connects the complete spectrum of a system, not only the sates with identical energy, like the symmetry algebra. More recently, however, a classical interpretation has also been introduced. Furthermore, it was conjectured that the transformations induced by the dynamical group of the Kepler problem can give the orbits of a planetary system. This question of Kepler is open for more that 400 years. In [9,17] we investigated whether or not some selected exo-planetary systems follow the symmetry-inspired regularity. Good agreement was found. In [28] we investigated all the known 100 systems with four or more planets. It turned out that the agreement between the observed data and the theoretical expectation is better than in our Solar system for 63 of them. We also pointed out that the successful but (so far) purely empirical (generalized) Titius-Bode rule follows from the transformations of the dynamical algebra.

It seems to be worthwhile to mention that in addition to the PhD thesis [13] 6 BSc thesis was prepared in relation with the research mentioned here, 4 TDK thesis, one of them won 2nd prize at OTDK, and the Dean's prize of the Debrecen University. Furthermore, the principle investigator have been invited to become a member of the topical editorial board of Symmetry.