Final report of NKFIH project No. FK 128673

Matroidal optimization

The project proposal addressed the major open problem of deciding the complexity of packing common bases in the intersection of two matroids. The research was motivated by a long list of graph and matroid optimization problems that can be formulated as special cases. This also implied that even if the problem proves to be hard, identifying special cases of the problem that are efficiently solvable is still of interest. In what follows, we give an overview of the main achievements of the project, together with some of the most important results.

1 Publications, conference and seminar talks, networking

During the funding period, the participants published almost 60 scientific papers, and two PhD theses will be submitted and defended by the young researchers of the project (Mendoza Cadena and Schwarcz).

Networking, work visits, and collaboration with external partners was an essential part of the project. The group hosted numerous short-term visitors from Germany, Switzerland, England, Japan, Poland, Sweden, and the United States. Furthermore, with the help of the Guest Scientist Fellowship of the Hungarian Academy of Sciences, we were able to invite Karthekeyan Chandraserkaran (University of Illinois) and his student for a long-term visit (3 months).

Most people from the group gave a talk on the 11th and 12th Hungarian-Japanese Symposium on Discrete Mathematics and its Applications; the PI, Frank and Kaszanitzky were invited speakers on the former one, and the PI was an invited speaker on the latter one as well. Csehi, Kaszanitzky, Király, and the PI were invited to the CIRM workshop Combinatorial Geometries: Matroids, Oriented Matroids and Applications where all of them gave a talk. The PI and Frank were invited to the 10th Cargese Workshop on Combinatorial Optimization where Frank gave talks. The PI was also an invited participant of the Dagstuhl Seminar 19211: Enumeration in Data Management. Kaszanitzky, Király and Tóthmérész were invited to give a talk on several mini-symposiums of the SIAM Conference on Applied Algebraic Geometry. Kaszanitzky was an invited speaker on the workshop Geometric constraint systems: rigidity, flexibility and applications. Bérczi attended the MATCH-UP 2022 conference in Vienna, where he presented a poster on the joint work with Csáji and Király about stable matchings, and also gave a talk on dynamic pricing schemes in combinatorial markets. Mendoza-Cadena gave a talk at the XXXIV. Hungarian Operations Research Conference on inverse arborescence problems with multiple cost functions. At the same event, Schwarcz gave a talk on rainbow and monochromatic circuits and cuts in binary matroids. Schwarcz also gave a talk on split matroids at the 29th British Combinatorial Conference in Lancaster, and a talk on reconfiguration of the union of arborescences at the ICALP 2023 conference. Bérczi gave talks on various topics at the Game Theory Seminar of the Corvinus University, the trimester Program on Discrete Optimization of the Hausdorff Research Institute for Mathematics, the Combinatorics Seminar of KTH Royal Institute of Technology, the Annual Meeting of the Canadian Society of Applied and Industrial Mathematics, the Bellairs Workshop on Discrete Optimization 2019, the Eleventh Cargese Workshop on Combinatorial Optimization in Cargese, the Boolean Seminar in Liblice. Most people from the group gave a talk on the 11th and the 12th Japanese-Hungarian Symposium on Discrete Mathematics and Its Applications.

The PI was the organizer of the twelth and fourteenth events of the Emléktábla Workshop Series. The goal of these workshops is to gather young mathematicians to work on problems posed by experienced senior researchers or by the participants themselves. The workshops took place in Gárdony (2022) and in Vác (2023) with the attendence of roughly 30 participants, including several researchers from abroad. As a result, several collaborations were formed.

It is worth emphasizing that, thanks in particular to this project, the PI applied succesfully to the Lendület grant of the Hungarian Academy of Science. Therefore, the work that was started within the framework of the current project is continued in the form of a research group on matroid optimization.

2 Matroids

The PI and Schwarcz proved that the problem of packing common bases in the intersection of two matroids is difficult under the rank oracle model, i.e., they showed that there is no algorithm which decides if the common ground set of two matroids can be partitioned into k common bases by using a polynomial number of independence queries. Their complexity result holds even for the very special case when $k = 2$, one of the matroids is a partition matroid, while the other matroid is linear and is given by an explicit representation. They also verified that the abstract problem of packing common bases in two matroids includes the NAE-SAT problem and the Perfect Even Factor problem in directed graphs. These results in turn imply that the problem is not only difficult in the independence oracle model but also includes NP-complete special cases.

The PI, Király and Csáji studied the special case of packing common bases when one of the matroids is a partition matroid while the other one is a graphic matroid. This setting is equivalent to the problem of packing rainbow spanning trees, an extension of the problem of packing arborescences in directed graphs which was answered by Edmonds' seminal result on disjoint arborescences. They complemented his result by showing that it is NP-complete to decide whether an edge-colored graph contains two disjoint rainbow spanning trees. The complexity result holds even for the very special case when the graph is the union of two spanning trees and each color class contains exactly two edges. As a corollary, they gave a negative answer to a question on the decomposition of oriented k-partition-connected digraphs.

Strongly base orderable matroids form a class for which a basis-exchange condition that is much stronger than the standard axiom is met. As a result, several problems that are open for arbitrary matroids can be solved for this class. In particular, Davies and McDiarmid showed that if both matroids are strongly base orderable, then the covering number of their intersection coincides with the maximum of their covering numbers. Motivated by their result, the PI and Schwarcz proposed relaxations of strongly base orderability in two directions. They weakened the basis-exchange condition, which led to the definition of a new, complete class of matroids with distinguished algorithmic properties. They also introduced the notion of covering the circuits of a matroid by a graph, and considered the cases when the graph is (A) 2-regular, or (B) a path. They gave an extensive list of results explaining how the proposed relaxations compare to existing conjectures and theorems on coverings by common independent sets.

The PI and Schwarcz investigated the existence of rainbow colored circuits and cuts in binary matroids. They showed that if a binary matroid of rank r is colored with exactly r colors, then the matroid either contains a rainbow colored circuit or a monochromatic cut. As the class of binary matroids is closed under taking duals, this immediately implies that, when colored by $n - r$ colors, the matroid either contains a rainbow colored cut or a monochromatic circuit. As a byproduct, they gave a characterization of binary matroids in terms of reductions to partition matroids.

Split matroids form a matroid class that was motivated by the study of matroid polytopes from a tropical geometry point of view. The PI, Király, Schwarcz, Yamaguchi and Yokoi provided a combinatorial study of this class. They introduce the notion of elementary split matroids, a subclass of split matroids that contains all connected split matroids. They gave a hypergraph characterization of elementary split matroids in terms of independent sets, and show that the proposed class is closed not only under duality and taking minors but also truncation.

The PI and Schwarcz studied the distance of basis pairs of a matroid in terms of symmetric exchanges. They gave an upper bound on the minimum number of exchanges needed to transform a basis pair into another for split matroids. As a corollary, they verified White's, Gabow's, Farber's, and Hamidoune's conjectures for this large class. Being a subclass of split matroids, their result settleed the conjectures for paving matroids as well.

Two pairs of disjoint bases of a matroid are called equivalent if one can be transformed into the other by a series of symmetric exchanges. In 1980, White conjectured that such a sequence always exists whenever the union of the basis-pairs are the same. A strengthening of the conjecture was proposed by Hamidoune, stating that minimum length of an exchange is at most the rank of the matroid. The PI, Mátravölgyi and Schwarcz proposed a weighted variant of Hamidoune's conjecture, where the weight of an exchange depends on the weights of the exchanged elements. They proved the conjecture for several matroid classes: strongly base orderable matroids, split matroids, graphic matroids of wheels, and spikes.

Together with Király, Yamaguchi and Yokoi, the PI studied how good a lexicographically maximal solution is in the weighted matching and matroid intersection problems. A solution is lexicographically maximal if it takes as many heaviest elements as possible, and subject to this, it takes as many second heaviest elements as possible, and so on. If the distinct weight values are sufficiently dispersed, e.g., the minimum ratio of two distinct weight values is at least the ground set size, then the lexicographical maximality and the usual weighted optimality are equivalent. They showed that the threshold of the ratio for this equivalence to hold is exactly 2. Furthermore, they proved that if the ratio is less than 2, say α, then a lexicographically maximal solution achieves $\alpha/2$-approximation, and this bound is tight.

The PI, Boros and Makino introduced a subclass of hypergraph Horn functions called matroid Horn functions. They provided multiple characterizations of matroid Horn functions in terms of their canonical and complete CNF representations. They also studied the Boolean minimization problem for this class, where the goal is to find a minimum size representation of a matroid Horn function given by a CNF representation. While there are various ways to measure the size of a CNF, they focused on the number of circuits and circuit clauses. They determined the size of an optimal representation for binary matroids, and gave lower and upper bounds in the uniform case. For uniform matroids, they showed a strong connection between the problem and Turán systems.

The PI, Mnich and Vincze considered the Bounded Degree g-polymatroid Element Problem with Multiplicities. Building on the approach of Király et al., they provided an algorithm for finding a solution of cost at most the optimum value, having the same additive approximation guarantee. As an application, they developed a 1.5-approximation for the metric Many-Visits TSP, where the goal is to find a minimum-cost tour that visits each city v a positive $r(v)$ number of times. As an extension of this result, they presented a 3/2-approximation algorithm for the metric many-visits path TSP problem, that can be seen as a far-reaching generalization of the 3/2-approximation algorithm for path TSP by Zenklusen.

3 Discrete Convex Analysis

Frank and Murota initiated a research that integrates the view of Edmond-type good characterizations and Discrete Convex Analysis. The considered problem is to find a lexicographically minimal integral vector in an integral base-polyhedron, where the components of a vector are arranged in a decreasing order. In a series of three papers, they gave a characterization of the optimal elements which lead to a strongly polynomial algorithm for the problem, derived min-max formulas as special cases of general discrete convex analysis results, and described the structure of decreasingly minimal integral feasible flows and developed a strongly polynomial algorithm for finding such a dec-min flow.

In another work, Frank and Murota proved a min-max formula for the minimum of an integer-valued separable discrete convex function where the minimum is taken over the set of integral elements of a box total dual integral (box-TDI) polyhedron. One variant of their theorem uses the notion of conjugate function (a fundamental concept in non-linear optimization) but they also provided another version that avoids conjugates, and its spirit is conceptually closer to the standard form of classic min-max theorems in combinatorial optimization. They presented a framework that provides a unified background for separable convex minimization over the set of integral elements of the intersection of two integral base-polyhedra, submodular flows, L-convex sets, and polyhedra defined by totally unimodular (TU) matrices. As an application, they showed how inverse combinatorial optimization problems can be covered by this new framework.

4 Rigidity

Király showed that, given any set of 26 generic points in the plane, one can always find an infinitesimally rigid realization of any rigid planar graph on that set. The proof can be extended for every class of graphs whose members can be embedded into a given closed surface. This result implies that there exists some constant c such that given any set A of $c\sqrt{n}$ generic points in the plane, one can find an infinitesimally rigid realization of any rigid graph on n vertices on A.

Kaszanitzky considered the logic puzzle Mainarizumu which is somewhat similar to the famous Sudoku game and presented a zero-knowledge protocol that can convince a verifier that a given instance of the puzzle has a solution and the prover also knows a solution without revealing any details of it. This is done by using physical objects, mostly cards. She also proved the NP-completeness of the puzzle.

Király and Mihálykó considered augmentation problems related to rigidity theory. For two integers $k > 0$ and ℓ, a graph $G = (V, E)$ is called (k, ℓ)-tight if $|E| = k|V| - \ell$ and $|E_G(X)| \leq k|X| - \ell$ for each $X \subseteq V$ that induces at least one edge. G is called (k, ℓ)-sparse if $G - e$ has a spanning (k, ℓ)-tight subgraph for all $e \in E$. They considered the problem of finding a graph $H = (V, F)$ with minimum number of edges such that $G + H$ is (k, ℓ)-sparse for a given graph $G = (V, E)$, and gave an $O(|V|^2)$ time algorithm and a min-max theorem for this augmentation problem when the input is (k, ℓ)-tight. They also showed that the components of the (k, ℓ)-sparsity matroid can be calculated in $O(|V|^2)$ time. This algorithm can be used to check global rigidity in several rigidity classes.

Kaszanitzky, Schulze and Tanigawa characterized the two dimensional global rigidity of infinite periodic frameworks under fixed lattice representations. Their combinatorial characterization of globally rigid generic periodic frameworks in the plain in particular implies toroidal and cylindrical counterparts.

Tanigawa showed that vertex-redundant rigidity of a graph implies its global rigidity in arbitrary dimension. Király, Kaszanitzky and Schulze extended this result to periodic graphs under fixed lattice representations. As an application of their result, they also gave a necessary and sufficient condition for the global rigidity of generic periodic body-bar frameworks in arbitrary dimension.

Király and Mihálykó considered the following augmentation problem: Given a rigid graph $G = (V, E)$, find a minimum cardinality edge set F such that the graph $G' = (V, E + F)$ is globally rigid. They provided a min-max theorem and a polynomial time algorithm for several types of rigidity, such as rigidity in the plane or on the cylinder.

Király with Garamvölgyi and Jordán considered two types of matroids defined on the edge set of a graph G: (k, ℓ)-count matroids, in which independence is defined by a sparsity count involving the parameters k and ℓ, and the (three-dimensional generic) cofactor matroid, in which independence is defined by linear independence in the cofactor matrix of G. They gave tight lower bounds, for each pair (k, ℓ), that show that if G is sufficiently highly connected, then $G - e$ has maximum rank for all $e \in E(G)$, and the (k, ℓ)-count matroid is connected. They also proved that if G is highly connected, then the vertical connectivity of the cofactor matroid is also high. They used these results to generalize Whitney's celebrated result on the graphic matroid of G to all count matroids and to the three-dimensional cofactor matroid.

5 Inverse problems

In minimum-cost inverse optimization problems, we are given a feasible solution to an underlying optimization problem together with a linear cost function, and the goal is to modify the costs by a small deviation vector so that the input solution becomes optimal. In 1998, Hu and Liu developed a strongly polynomial algorithm for solving the inverse arborescence problem that aims at modifying minimally a given cost function on the edge-set of a digraph so that an input arborescence becomes a cheapest one. Frank and Hajdu developed a conceptually simpler algorithm along with a min-max theorem for the minimum modification of the cost-function. The approach is based on a link to a min-max theorem and a two-phase greedy algorithm by Frank from 1979 concerning the primal optimization problem of finding a cheapest subgraph of a digraph that covers an intersecting family along with the corresponding dual optimization problem, as well.

The PI, Mendoza-Cadena and Varga introduced a new class of inverse optimization problems in which an input solution is given together with k linear weight functions, and the goal is to modify the weights by the same deviation vector p so that the input solution becomes optimal with respect to each of them, while minimizing $\|p\|_1$. In particular, they concentrated on three problems with multiple weight functions: the inverse shortest $s-t$ path, the inverse bipartite perfect matching, and the inverse arborescence problems. Using LP duality, they gave min-max characterizations for the ℓ_1-norm of an optimal deviation vector. Furthermore, they showed that the optimal p is not necessarily integral even when the weight functions are so, therefore computing an optimal solution is significantly more difficult than for the single-weighted case. They gave a necessary and sufficient condition for the existence of an optimal deviation vector that changes the values only on the elements of the input solution, thus giving a unified understanding of previous results on arborescences and matchings.

The difference between the new and the original cost functions can be measured in several ways. The PI, Mendoza-Cadena and Varga focused on two objectives: the weighted bottleneck Hamming distance and the weighted ℓ_∞-norm. They considered a general model in which the coordinates of the deviation vector are required to fall within given lower and upper bounds. For the weighted bottleneck Hamming distance objective, they presented a simple, purely combinatorial algorithm that determines an optimal deviation vector in strongly polynomial time. For the weighted ℓ_∞-norm objective, they gave a min-max characterization for the optimal solution, and provided a pseudo-polynomial algorithm for finding an optimal deviation vector that runs in strongly polynomial time in the case of unit weights.

6 Dynamic pricing

The PI, Kakimura and Kobayashi studied the problem of maximizing social welfare in combinatorial markets through pricing schemes. They considered the existence of prices that are capable to achieve optimal social welfare without a central tie-breaking coordinator. In the case of two buyers with rank valuations, they gave polynomial-time algorithms that always find such prices when one of the matroids is a simple partition matroid or both matroids are strongly base orderable. This result partially answered a question raised by Düetting and Végh in 2017.

The PI, Bérczi-Kovács and Szögi considered dynamic pricing schemes in combinatorial markets. By relying on an optimal dual solution, they showed the existence of optimal dynamic prices in unit-demand markets and in multi-demand markets up to three buyers, thus giving new interpretations of results of Cohen-Addad et al. and Berger et al., respectively. They further provided an optimal dynamic pricing scheme for bi-demand valuations with an arbitrary number of buyers.

Though the dynamic setting is capable of maximizing social welfare in various scenarios, the assumption that the agents arrive one after the other eliminates the standard concept of fairness. The PI, Codazzi, Golak and Grigoriev studied the existence of optimal dynamic prices under fairness constraints in unit-demand markets. They proposed four possible notions of envy-freeness of different strength depending on the time period over which agents compare themselves to others: the entire time horizon, only the past, only the future, or only the present. For social welfare maximization, while the first definition leads to Walrasian equilibria, they gave polynomial-time algorithms that always find envy-free optimal dynamic prices in the remaining three cases. In contrast, for revenue maximization, the showed that the corresponding problems are APX-hard if the ordering of the agents is fixed.

7 Further results

On arborescence packings, Király gave some new results with Szigeti and Tanigawa. They considered a paper by Edmonds where he observed a characterization of packing spanning arborescences in terms of matroid intersection. They extended this characterization to the setting of reachability-based packing of arborescences which was introduced by Király as a common generalization of two different extensions of Edmonds' problem the problem of packing reachability arborescences by Katoh et al. and the problem of matroid-based packing of arborescences by Durand de Gevigney et al.. This new characterization of the arc sets of reachability-based packing in terms of matroid intersection gave an efficient algorithm for the minimum weight reachability-based packing problem.

The PI, Chadrasekaran, Király, and Madan investigated the approximability of the so-called linear 3-cut problem in directed graphs. The problem itself is NP-hard, and the main question was its approximability. Prior to their work, the best known lower bound was 4/3 (assuming the unique games conjecture) while the best known upper bound was 2 using a trivial algorithm. They completely closed the gap by presenting a $\sqrt{(}2)$-approximation algorithm and showing that this factor is tight under UGC.

The fixed-terminal bicut and global bicut problems are natural extensions of the $\{s,t\}$-min cut and global min-cut problems, respectively, from undirected graphs to directed graphs. Fixed-terminal bicut is NP-hard, admits a simple 2-approximation, and does not admit a $(2-\varepsilon)$-approximation for any constant $\varepsilon > 0$ assuming the unique games conjecture. The PI, Chandrasekaran, Király, Lee, and Xu showed that global bicut admits a $(2-1/448)$-approximation, thus improving on the approximability of the global variant in comparison to the fixed-terminal variant.

In the multiway cut problem, we are given an undirected graph with non-negative edge weights and a collection of k terminal nodes, and the goal is to partition the node set of the graph into k non-empty parts each containing exactly one terminal, so that the total weight of the edges crossing the partition is minimized. The problem is APX-hard for $k \geq 3$, the best known inapproximability result being 1.2. The PI, Chandrasekaran, Király and Madan improved the lower bound to 1.20016 under UGC.

The Bernardi action is an action of the sandpile group of a graph on the spanning trees of the graph. This action depends on an embedding of the graph into a surface and on a fixes base vertex. Baker and Wang showed that the ection is independent of the base vertex if and only if the embedding is into the plane. Kálmán, Lee and Tóthmérész show the “if” part of the above statement by giving a canonical definition for the planar Bernardi action.

Kálmán and Tóthmérész associated root polytopes to directed graphs and studied them by using ribbon structures. Given a ribbon structure, they identified a natural class of spanning trees and showed that, in the semi-balanced case, they induce a shellable dissection of the root polytope into maximal simplices. They obtained a general recursion relation, and worked out the case of layer-complete directed graphs, where their method recovers a previously known triangulation.