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ALGEBRAS AND THEIR CLASSES: STRUCTURE AND COMPLEXITY

Researchers in the project

2018–2022: Maróti, Miklós; Szendrei, Ágnes; B. Szendrei, Mária (principal investigator); Tóth, Endre
2018–2021: Szakács Nóra
2021–2022: Gyenizse, Gergő; Kunos, Ádám; Waldhauser, Tamás

SUMMARY OF SCIENTIFIC RESULTS

Universal algebra

During the past decades, the commutator theory developed by Freese and McKenzie [FM.1987] has become a basic tool in the investigation of congruence modular (= CM) varieties — generalizing the most familiar classes of algebras like groups, rings, lattices, and Boolean algebras —, and has had enormous impact far beyond. Kiss [Kis.1992] discovered 4-ary difference terms and their significance in characterizing the commutator in CM varieties. Many non-CM varieties also have difference terms, e.g., the variety of semilattices. In **[16]** we extend Kiss' results to all varieties with a difference term. In **[9]** we apply techniques for the modular commutator to classify the structure of finite algebras with a cube term which satisfy a strong version of the "split centralizer condition" introduced by the first and third authors to prove a general dualizability theorem.

Supernilpotence, introduced by Aichinger and Mudrinski [AM.2010], is a more recent notion than the commutator, and is based on higher arity versions of the commutator. In **[3]** we prove that for finite algebras, supernilpotence implies nilpotence. Examples constructed by Moore–Moorhead (2019) and Weinell (2019) show that this implication may fail for infinite algebras. The paper **[11]** contains two algorithmic results related to supernilpotence for finite algebras A: (1) If the variety $\mathbf{V}(A)$ generated by A omits type 1, then it is decidable whether a given congruence of A is supernilpotent. (2) If $\mathbf{V}(A)$ has a cube term and for every subdirectly irreducible section S of A the centralizer of the monolith of S is supernilpotent, then there exists a polynomial time algorithm for the Subpower Membership Problem for A. Our argument reduces the proofs of (1)–(2) to previously known special cases: $(1)^*$ the question is whether A is supernilpotent; $(2)^*$ A is assumed to be supernilpotent. For the reduction we use the composition of two known functors to "make algebras from congruences", and show that this construction preserves a lot of algebraic properties.

[8] and **[22]** are the first two papers of an ongoing project on classifying all abelian minimal varieties. The locally finite case was settled about 25 years ago, using tools from commutator theory, tame congruence theory, and clone theory. The main result of **[8]** is that the dichotomy known in the locally finite case holds in general: Every non-affine abelian variety contains a nontrivial, strongly abelian subvariety. Hence, every minimal abelian variety is either affine or strongly abelian. Paper **[22]** is about affine varieties, and shows that the category of affine varieties, up to term equivalence, is categorically equivalent to the category of modules, considered as 2-sorted structures, with an additional unary operation from the "module sort" to the "ring sort".

In **[10]** we introduce a new closure operator, called "ultralocal closure", on the clone lattice on an infinite set and characterize the ultralocal closure of a clone. We deduce that every clone containing a near unanimity operation is ultralocally closed (our version of Vaggione's theorem in [V.2018]) and so is the clone of every simple module. In **[7]** we prove that every minimal dual discriminator variety is minimal as a quasivariety. In **[1]** the class of commutative unital rings in which no two distinct elements divide the same elements is characterized in terms of integral domains.

In [KMZ.2019] the authors presented a new family of finite bounded posets whose clones of monotone operations are not finitely generated. The proofs of this result, as well as earlier results of the third author, are all relying on the description of obstructions of these relational clones and an ingenious construction introduced by Tardos in his famous paper [T.1986]. However, describing the structure of all obstructions has proven to be a very difficult task for certain bounded posets. The natural continuation of these investigations was the study of the clone of locked crowns where the crown has at least six elements. Unfortunately the obstructions even for the crowns are unknown, although we expect that they will be building blocks for the obstructions of the locked crowns. It is well known that every critical relation of a relation clone has at least one corresponding obstruction, but not all obstructions give rise to a critical relation. We have observed that, for some posets, describing the critical relations is easier than describing all obstructions, but for the Tardos poset the opposite seems to be the case. In **[17]** we describe the critical relations of crowns, and as a corollary, give a polynomial time algorithm to the subpower membership problem for any crown.

We say that a variety **V** interprets in **W**, if we can uniformly replace the basic operation symbols of **V** with terms of **W** such that the identities of **V** become identities of **W**. For example the variety of semilattices is interpretable in **W** if and only if **W** has a binary term operation that satisfies the semilattice axioms in every algebra of **W**. The class of varieties factored by equi-interpretability form a lattice, called the lattice of interpretability types. This lattice, introduced by Garcia and Taylor in [GT.1984], is arguably the most important structure in the subfield of Mal'tsev conditions, but it is still poorly understood. They conjectured that the Mal'tsev condition characterizing congruence 2-permutability defines a join-prime element in this lattice. We have investigated this, and some related problems in a series of papers. Let **M** be the variety of algebras defined by the majority Mal'tsev condition $m(x,x,y) = m(x,y,x) = m(y,x,x) = x$. In **[2]** we present a finitely generated, congruence non-5-permutable variety **P** such that its join with the variety **M** is congruence 5-permutable. This shows that the variety defined by the congruence 5-permutability Mal'tsev condition, or n-permutability for n at least 5, is not a join-prime element in the lattice of interpretability types. In **[14]** we show that a similar approach cannot be used to prove non-primeness for congruence 3-permutability. Specifically, we prove that every locally finite non-3-permutable variety interprets in a non-3-permutable variety with a majority operation. However, this result leaves open whether congruence 3-permutability is join-prime in the lattice of interpretability types. In the recent paper **[15]** we prove the Taylor–Garcia conjecture by applying some nontrivial combinatorial properties of graph exponentiation.

Ježek and McKenzie [JM.2009i,iii,iv, JM.2010ii] published a series of four papers that dealt with definability in substructure ordering of semilattices, ordered sets, distributive lattices, and lattices. The automorphism groups of the substructure ordering of these structures turned out to be isomorphic to either the trivial or the two-element group. This pattern was broken in [Ku.2021] where definability in the substructure ordering of finite directed graphs was investigated. The automorphism group was proved to be a finite group having a specific 768-element subgroup, and the author conjectured that this subgroup coincides with the automorphism group. In **[21]**, we reinforce this conjecture by showing that the automorphism group behaves on the first few levels as expected. With the use of computer calculation, we analyze the first four levels of the ordering in question, holding 3160 directed graphs. We show that the poset of the first three levels has approximately 1.67×10^{13} automorphisms out of which only 768 extend to the whole ordering — exactly those indicated by the conjecture. This is a proper solidification of the conjecture, though far from being a proof.

For an arbitrary finite algebra A, we investigate the question which sets of n-tuples appear as solution sets of systems of equations over A. These sets are analogues of algebraic varieties investigated in algebraic geometry, and thus their study might be regarded as universal algebraic geometry (see [P.2007]). We say that an algebra A has property (SDC) if the family of all sets of n-tuples that are closed under the centralizer of A coincides with the family of all sets that appear as solution sets of systems of equations over A. Here "SDC" stands for "Solution sets are Definable by the Centralizer".

All two-element algebras were proved in [TW.2017] to have this property. We prove in **[5]** that the centralizer is the only clone that can possibly describe solution sets, and we also characterize this property in terms of a certain quantifier elimination property. We use this characterization to completely describe finite lattices and semilattices having property (SDC): a finite lattice has property (SDC) if and only if it is a Boolean lattice, and a finite semilattice has property (SDC) if and only if it is the semilattice reduct of a finite distributive lattice. In **[12]** we obtain a description of the centralizers of arbitrary finite distributive lattices and semilattices, which gives an insight into the form of the n-ary operations belonging to the centralizer of an arbitrary finite lattice, and it is useful in counting the essentially n-ary operations belonging to the centralizers of finite lattices and semilattices. Since a finite relational structure has quantifier elimination for primitive positive formulas if and only if it is polymorphism-homogeneous (see [PP.2015]), property (SDC) of an algebra A is equivalent to the polymorphism-homogeneity of a certain relational structure derived from A. In **[20]** we prove that this is also equivalent to polymorphism-homogeneity of the algebra A itself. In **[26]** polymorphism-homogeneity of three-element groupoids is studied. A computer program developed by the author is used to find non-extendable local polymorphisms, and by means of them, it is determined for more than 300 out of the 411 groupoids (up to clone equivalence) whether they are polymorphism-homogeneous or not.

There are several ways to measure how far a binary operation (or a groupoid) is from being associative. One of them is the associative spectrum introduced in [CsW.2000]. The associative spectrum is a sequence of natural numbers whose n-th member, roughly speaking, counts how many of the "bracketing identities" implied by associativity are not satisfied by the groupoid. Thus, the larger the spectrum, the less associative the groupoid is. Quasigroups, being nonassociative generalizations of groups, seem worthwhile to study from this point of view. A quasigroup is called linear if its multiplication is of the form $xy = \varphi(x) + \psi(y)$ where + is a group operation (not necessarily commutative), and φ, ψ are automorphisms of this group. In **[23]** we investigate associative spectra of such linear quasigroups. Our main result is a complete characterization of the associative spectra of linear quasigroups. However, determining the exact numerical values of these spectra is a nontrivial counting problem even in the simplest special cases (e.g., when $\varphi = \psi$), and our computer calculations gave many sequences that are not yet included in the On-Line Encyclopedia of Integer Sequences.

Semigroups

Inverse semigroups are one of the most important generalizations of groups, that originally emerged as the abstract counterparts of partial symmetries.

An inverse monoid (= semigroup with identity element) is F-inverse if each of its classes modulo the least group congruence σ has a maximum element with respect to the natural partial order. Thus every F-inverse monoid can be equipped with the unary operation $a \to a^{\mathbf{m}}$ where $a^{\mathbf{m}}$ denotes the maximum element in the σ-class of a. It was noticed by Kinyon [Kin.2018] that the class of all F-inverse monoids forms a variety in this enriched signature. In **[6]** we describe the universal objects in several classes of F-inverse monoids, in particular, the free F-inverse monoids. More precisely, for every X-generated group G we provide a transparent model for the initial object in the category of all X-generated F-inverse monoids F for which $F/\sigma = G$. This object is a common generalization of the Margolis–Meakin expansion and the Birget–Rhodes expansion of G.

Since usual semidirect products of inverse semigroups fail to be inverse in general except when the second factor is a group, Billhardt [B.1992] introduced a modified version of this construction called λ-semidirect product, and proved a Kaloužnin–Krasner-type theorem, that is, proved that any extension of inverse semigroups defined by a kind of congruence named after him by Lawson [L.1998] is embeddable in a λ-semidirect product. Moreover, a subclass of Billhardt congruences called split was also introduced, and the extensions via such congruences were embedded into a full restricted semidirect product which is an inverse subsemigroup of the respective λ-semidirect product provided the action fulfils additional conditions. In **[24]** we prove stronger results than those in [B.1992] for a more general class of extensions. More precisely, we define a family of congruences, called almost

Billhardt, which generalize Billhardt congruences, and prove that (1) the extensions isomorphic to full restricted semidirect products are, up to isomorphism, those defined by split almost Billhardt congruences, and (2) each extension defined by an almost Billhardt congruence θ is embeddable in a full restricted semidirect product such that the components of the first factor are direct products of idempotent θ-classes. This refutes the general view, formulated also in [L.1998, p. 156], that Billhardt congruences are intimately connected to λ-semidirect products and split Billhard congruences to full restricted semidirect products. Since the kernel of a λ-semidirect product is "much larger" then the first factor, the question arises whether there exists a generalized version of a full restricted semidirect product such that its kernel is isomorphic to the first factor but each extension "weakly" embeddable in a λ-semidirect product is embeddable in such a new product. In **[25]** we introduce such a construction.

Garrão, Martins-Ferreira, Raposo and Sobral [GMRS.2020] introduced the notions of a conjugation semigroup and monoid, and studied the category of cancellative conjugation monoids in order to present a new weakly Mal'tsev category that fails to be Mal'tsev. Thus, roughly speaking, the paper is about Schreier-type extensions of such monoids. Conjugation is an additional unary operation on a semigroup such that the unary semigroup obtained — and called a conjugation semigroup — satisfies certain identities stemming from conjugation on the multiplicative semigroup of quaternions. It is natural to ask "how far" cancellative conjugation monoids — and more generally, cancellative conjugation semigroups — are from groups. It is shown in **[18]** that the conjugations of a group are in one-to-one correspondence with the endomorphisms of the group whose ranges are in the center. Moreover, cancellative conjugation semigroups are proved to be, up to isomorphism, the conjugation subsemigroups of conjugation groups.

Hyperbolic groups have been in the center of geometric group theory for a long time, and the results of this area have great influence also in other classes of algebras. In **[13]** we introduce a new approach to the study of hyperbolicity and related geometric conditions in inverse monoids. In the class of inverse monoids, the role of the Cayley graph of a group is taken over by the strongly connected components of the Cayley graph, called Schützenberger graphs. The broadest class of inverse monoids that one might hope to establish good algorithmic properties for in general is the class of finitely presented inverse monoids with hyperbolic Schützenberger graphs. However, we show that this class has a member with an unsolvable word problem. On the other hand, we prove that the class of finitely presented inverse monoids whose Schützenberger graphs are quasi-isometric to trees has a uniformly solvable word problem, and furthermore, the languages of their Schützenberger automata are context-free.

Paper **[4]** gives an overview of the most substantial developments of the last 50 years in the structure theory of regular semigroups. A selection of results of Mario Petrich that have had fundamental influence on the development of the theory of semigroups, in particular, of inverse and of completely regular semigroups are reviewed in **[19]**. In both cases, the author was invited by the editorial board of Semigroup Forum to write these articles.

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ADDENDUM TO THE FINANCIAL REPORT
To whom it may concern

I find it misleading that the list of the travel costs does not contain information on the fact that G. Gyenizse was not able to attend on the meeting Model Theory and Applications 2022 (Universitá de della Campanilla L.Vanvitelli, Cetraro, Italy). His Wizz Air flight from Budapest to Rome was canceled several hours before the departure time, and the airline has not answered his application for reimbursement for almost 100 days.