PROJECT CLOSING REPORT (FINAL REPORT) OF KH 126581

Technical notes. The ***list*** of scientific papers, referenced often in this report, is at the end of this file. At some items in the *list*, there is a second bracketed number or [*] referring to

`http://www.math.u-szeged.hu/~czedli/listak/publist.html` ,

with further links in some cases. Nine papers in the *list* have already appeared as Open Access; they are marked by ***OA***. The rest of the papers in the *list* are available from free repositories in preprint form (and [10], [11], [15], and [18] is expected to appear as ***OA***). All the journals occurring in the *list*, that is, all journals where [1]–[18] have appeared or have been submitted to, are reviewed by MathSciNet. Thirteen of [1]–[18] have been supported exclusively by KH 126581, and eight of these thirteen is ***OA***.

Excerpts from the original Research Plan. The original Research Plan, which was one of the documents submitted in order to receive support for the KH 126581 project, contained the following (in Hungarian): "*We plan to submit at least six research papers to MathSciNet-reviewed journals in the whole (two year long) project, and at least three of these papers will be written in the first year. Also, the computer program mentioned in Section 3*[1] *will be made in the first year.*"

Computer programs developed with the support of the Project. The computer program mentioned in the Research Plan was developed in the first year of the Project, and it is freely available from G. Czédli's website,

`http://www.math.u-szeged.hu/~czedli/` ;

one can find it under the name `wpflCzG` after clicking on "Computer programs" in the left panel. In the second year of the Project, two additional freely downloadable computer programs were also created; they are called `subsize` and `sublatts`, and they can be found on the same website. These two programs are not independent; `sublatts` was obtained from `subsize` by adding a useful new feature to it that takes the peculiarities of lattices into account. These three programs run under Windows 10 environment, and each of them acknowledges the support of the KH 126581 Project. The use of these programs will be explained where the corresponding research papers are described. Further applications of (at least some of) these three programs in the future are strongly expected.

Utilization of the results. The original Research Plan claimed that the results of the project would be exploited in the training of (Hungarian and foreign) PhD students. In full accordance with this claim, [2], [14], [15], and [17] are joint with two current PhD students of the Doctoral School in Mathematics, Szeged. One of these PhD students is a foreign student supported by Stipendium Hungaricum while the other one is Hungarian. The topics developed within the KH 126581 Project will play important roles in their forthcoming PhD dissertations. Since the

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[1] i.e., Section 3 of the Research Plan

result (but not the whole proof) of [2] can easily be understood at high school level, it might be used to popularize mathematics taught at the University of Szeged.

The description of the results achieved. For the rest of this document, n will denote a positive integer number.

• **1.** By a twenty year old result of Ralph Freese, an n-element lattice L has at most 2^{n-1} congruences. In Czédli [1], we generalize this result for every congruence-distributive n-element algebra and, in addition, we show that if such an algebra has 2^{n-1} congruences then its congruence lattice is boolean. For the particular case of lattices, which are well known to be congruence distributive, [1] proves that if an n-element lattice has less than 2^{n-1} congruences, then it has at most 2^{n-2} congruences, and we describe the n-element lattices with exactly 2^{n-2} congruences. Note that the investigations started in [1] have already been continued by Júlia Kulin and Claudia Mureşan in `https://doi.org/10.1007/s11083-019-09514-2` (see also `https://arxiv.org/abs/1801.05282`). The results of [1] harmonize with Section 6 "Study of special lattices" of the Research Plan.

• **2.** In [2], where the first author is a PhD student of the third one, an n-sided polygon lying on a circular arc or, shortly, an *n-fan* is understood to be a sequence of $n + 1$ points on a circle going counterclockwise such that the total rotation δ from the first point to the last one is at most 2π (in another scale, 360°). We prove that for $n \geq 3$, the n-fan cannot be constructed with straightedge and compass in general from its central angle δ and its central distances, which are the distances of the edges from the center of the circle. Also, we prove that for each fixed δ in the interval $(0, 2\pi]$ and for every $n \geq 5$, there exists a concrete n-fan with central angle δ that is not constructible from its central distances and δ. The present paper generalizes some earlier results published by the second author and Á. Kunos on the particular cases $\delta = \pi$ (Thalesian polygons) and $\delta = 2\pi$ (inscribed polygons). This paper harmonizes slightly with the Research Plan, which aims a bit also at geometrical questions.

• **3.** Paper [3] is a "semilattice theoretical counterpart" of the lattice theoretical paper [1] and Júlia Kulin and Claudia Mureşan's above-mentioned paper. Let NCSL(n) denote the set of sizes of congruence lattices of n-element semilattices. In [3], we have found the four largest numbers belonging to NCSL(n), provided that n is large enough to ensure that $|\text{NCSL}(n)| \geq 4$. These four numbers are $32 \cdot 2^{n-6}$, $28 \cdot 2^{n-6}$, $26 \cdot 2^{n-6}$, and $25 \cdot 2^{n-6}$. Furthermore, we have described the n-element semilattices witnessing these numbers. The results of [3] harmonize with the "questions motivated by lattice theory" part of the title of the project.

• **4.** K. Adaricheva and M. Bolat have recently proved that if U and V are circles in a triangle, then we can select one of the two circles and two of the three vertices of the triangle such that the convex hull of the three selected objects includes the non-selected circle. One could say disks instead of circles. Generalizing this result, [4] proves the same for any two compact sets U and V in a triangle such that V is obtained from U by a positive homothety or by a translation. Also, we give a survey in this paper to explain how lattice theoretical antecedents lead to this geometrical result. Paper [4] belongs to convex geometry, which has been highlighted in Section 1 of the Research Plan.

• **5.** Paper [5] fits perfectly in Section 3 "A program for the word problem of lattices" of the Research Plan, and it is a paper that has benefited from the first computer program, called `wpflCzG` and mentioned earlier. By a 1941 result of Ph. M. Whitman, the free lattice FL(3) on three generators includes a sublattice S that is isomorphic to the lattice FL(ω) generated freely by denumerably many elements. In his 2016 paper, G. Czédli symmetrized this classical result by constructing a sublattice $S \cong \mathrm{FL}(\omega)$ of FL(3) such that S is selfdually positioned in $FL(3)$ in the sense that it is invariant under the natural dual automorphism of FL(3) that keeps each of the three free generators fixed. Now, [5] moves to the furthest in terms of symmetry by constructing a selfdually positioned sublattice $S \cong \mathrm{FL}(\omega)$ of FL(3) such that every element of S is fixed by all automorphisms of FL(3). That is, in our terminology, [5] embeds FL(ω) into FL(3) in a *totally symmetric way.* The main result of [5] determines all pairs (κ, λ) of cardinals greater than 2 such that FL(κ) is embeddable into FL(λ) in a totally symmetric way. Also, we relax the stipulations on $S \cong \mathrm{FL}(\kappa)$ by requiring only that the set S (rather than each of its elements) is closed with respect to the automorphisms of FL(λ), or S is selfdually positioned and closed with respect to the automorphisms; we determine the corresponding pairs (κ, λ) even in these two cases.

• **6.** Paper [6], which is motivated by [1], proves that if an n-element finite lattice L has more than 2^{n-5} congruences, then L is planar. This result is sharp, since for each $n \geq 8$, there exists an n-element non-planar lattice with exactly 2^{n-5} congruences. This result harmonizes with Section 6 "study of special lattices" of the Research Plan.

• **7.** Similarly to [4], paper [7] also belongs to convex geometry, which has been highlighted in Section 1 of the Research Plan. Let U be a compact convex subset of the plane, and assume that whenever a subset V of the plane is congruent to U, then U and V are not crossing in a natural sense due to L. Fejes-Tóth. In other words, at least one of the sets $U \setminus V$ and $V \setminus U$ is (path-)connected if V is congruent to U. A theorem of L. Fejes-Tóth from 1967 states that the assumption above holds for U if and only if U is a disk. In a paper appeared in 2017, the author introduced a new concept of crossing, and proved that L. Fejes-Tóth's theorem remains true if the old concept is replaced by the new one. The purpose of [7] is to describe the hierarchy among several variants of the new concept and the old concept of crossing. In particular, [7] proves that each variant of the new concept of crossing is more restrictive then the old one. Therefore, L. Fejes-Tóth's theorem from 1967 becomes an immediate consequence of the 2017 characterization of circles but not conversely. Finally, a mini-survey shows that this purely geometric paper has precursors in combinatorics and, mainly, in lattice theory.

• **8.** Paper [8] proves that if an n-element lattice has at least $83 \cdot 2^{n-8}$ (nonempty) sublattices, then this lattice is planar. For $n > 8$, this result is sharp since there exists an n-element non-planar lattice with exactly $83 \cdot 2^{n-8} - 1$ sublattices. In addition to the usual amount of lattice theoretical arguments, the proof requires the brutal force of a desktop computer; this is why the second computer program, called `subsize` and mentioned above, was developed. The extended arXiv version of the paper shows how this program was used. The result harmonizes with Section 6 "study of special lattices" of the Research Plan.

• **9.** Combining the use of the above-mentioned program `subsize` with theoretical arguments again, [9] proves that if an n-element semilattice S has at least $127 \cdot 2^{n-8}$ (nonempty) subsemilattices, then S is planar. For $n > 8$, this result is sharp since there exists an n-element non-planar semilattice with exactly $127 \cdot 2^{n-8} - 1$ subsemilattices. The result of [9] harmonizes with the "questions motivated by lattice theory" part of the title of the project.

• **10.** Finite (upper) nearlattices are essentially the same mathematical entities as finite semilattices, finite commutative idempotent semigroups, finite join-enriched meet semilattices, and chopped lattices. For definition, let $(L; \vee)$ be a finite join-semilattice. Its natural ordering is defined by $x \leq y \iff x \vee y = y$. For $x, y \in L$, let $x \wedge y$ be the infimum of $\{x, y\}$ provided this infimum exists, and let $x \wedge y$ be undefined otherwise. Finite nearlattices are the structures $(L; \vee, \wedge)$ obtained in this way. A nonempty subset S of L is called a *subnearlattice* of $(L; \vee, \wedge)$ if S is $\vee$-closed and for every $x, y \in S$ such that $x \wedge y$ exists in L, we have that $x \wedge y \in S$. In [10], we prove that if an n-element nearlattice L has at least $83 \cdot 2^{n-8}$ subnearlattices, then L is planar; the result is sharp for $n > 8$. The proof includes long theoretical arguments, a substantial use of [8], and our third computer program, called `sublatts` and mentioned above. Note that the program `subsize` would hardly have been efficient enough for this purpose. The extended arXiv version of [10] shows how `sublatts` was used. The result of [10] harmonizes with the "questions motivated by lattice theory" part of the title of the project.

• **11.** For an n-tuple $x = (x_1, ..., x_n)$ of elements in a lattice L, an element $m \in L$ is a *median* of x if (in the Hasse diagram of L) the sum of path distances of m from the elements x_i ($i = 1, ..., n$) is minimal. Extending the 2016 result of the same authors, [11] proves that any median of x is less than or equal to $x_1 \vee \cdots \vee x_n$ in a finite semimodular lattice of breadth 2; this result is sharp in the sense that breadth 2 cannot be replaced by breadth 3. The result harmonizes with Section 6 "Study of special lattices" of the Research Plan.

• **12.** For a given n and a class $\mathcal{V}$ of algebras, let $\mathrm{SubS}(n, \mathcal{V})$ denote the set of possible numbers of subalgebras (including the emptyset) of n-element algebras in $\mathcal{V}$.

Paper [12] determines the three largest numbers in $\mathrm{SubS}(n, \{\text{lattices}\})$ for $n \geq 5$; these numbers are $32 \cdot 2^{n-5} = 2^n$, $26 \cdot 2^{n-5}$, and $23 \cdot 2^{n-5}$. Also, [12] describes the lattices that give rise to these numbers. These results harmonize with Section 6 "Study of special lattices" of the Research Plan.

• **13.** For definition, recall that when posets are considered categories, then isotone maps and their residual maps correspond to functors and their right adjoints, respectively. That is, if A and B are posets and $u \colon A \to B$ and $v \colon B \to A$ are functions (that is, maps), then v is said to be the *residual* of u iff for every pair $(x, y) \in A \times B$, $u(x) \leq y \iff x \leq v(y)$. The residual of u is unique provided it exists, and functions with residuals are necessarily isotone. For a function f from a set X to a lattice L, i.e., for a *lattice-valued function* $f \colon X \to L$, the f-preimages of principal filters of L are called the *cuts* of f. In [13], the authors prove that in case of a lattice-valued function f, if the codomain lattice L is complete, then f induces a residuated map from L to the powerset $P(X)$ of the domain set X of f. For general poset-valued (rather than just lattice-valued) functions, they give conditions under which the map sending an element to the corresponding cut is a quasi-residuated function, and then conditions under which this function is residuated; several cases are discussed.

• **14.** Paper [14], where the first author is a PhD student of the second one, continues the investigations of [12] and also harmonizes with the Research Plan. The authors of [14] show that the fourth and fifth largest numbers in $\mathrm{SubS}(n, \{\text{lattices}\})$, defined in (• 12), are $21.5 \cdot 2^{n-5}$ for $n \geq 6$ and $21.25 \cdot 2^{n-5}$ for $n \geq 7$, respectively, and they describe the lattices that give rise to these numbers.

• **15.** We say that a function f from a finite algebra A to itself is in the *centralizer clone* (or, in short, the *centralizer*) of A if f commutes with all term functions (equivalently, with all operations) of A. The above-mentioned f will act on tuples componentwise. A finite algebra A has Property (SDC) if for every n, the sets of n-tuples closed with respect to every f in the centralizer of A are exactly the solution sets of systems of equations on n indeterminates. Paper [15], where the first author is a PhD student of the second one, proves that a finite lattice L and a finite semilattice S have (SDC) if and only if L is boolean and S is distributive. This result harmonizes with Section 2 "Study of functions defined on lattices" of the Research Plan.

• **16.** With join and meet acting as addition and multiplication, the set $M_n(L)$ of n-by-n matrices over a distributive lattice L with respect to the usual matrix multiplication is known to be an associative groupoid, i.e., a semigroup. It is proved in [16] that for a non-distributive lattice L, the groupoid $M_n(L)$ is as far from being associative as it is possible in terms of associative spectra; this means that for every n, the possible bracketings of an n-fold product $x_1 x_2 \cdots x_n$ yield pairwise different term functions. The invertible matrices are also determined if 0 or 1 is irreducible in L. Matrix multiplication defined above is a function defined with the help of a lattice, so the results of [16] harmonize with Section 2 "Study of functions defined on lattices" of the Research Plan.

• **17.** In addition to [15], several other results on centralizers of lattices and semilattices are proved in [17], where the first author is a PhD student of the second one. While a lattice term, even for a two-element lattice, can be of arbitrarily large essential arity, it is a surprising result of [17] that for any finite lattice, there is an upper bound on the essential arities of operations in the centralizer. For distributive lattices, the least upper bound is determined in terms of Boolean sublattices and quotient lattices. Considering only the join (or the meet) operation of a finite lattice, the centralizer becomes much larger: the number of essentially n-ary operations in the centralizer is shown to grow exponentially. The exact numbers are determined for finite chains. These results harmonize with Section 2 "Study of functions defined on lattices" of the Research Plan.

• **18.** For a lattice L, let $\mathrm{At}(L)$ and $\mathrm{CoAt}(L)$ denote the set of atoms and the set of coatoms of L, respectively. Shortly before the end the second year of the project, our experience with free lattices, including [5], a 2015 result available at `http://dx.doi.org/10.14232/actasm-015-586-2`, and all 3-generated lattices we had ever seen in the literature, raised the following problem:

$$\text{are } |\mathrm{At}(L)| \text{ and } |\mathrm{CoAt}(L)| \text{ in } \{1, 2, 3\} \text{ for every 3-generated lattice } L? \qquad (1)$$

In order to deal with this problem, we asked for and we was given a month prolongation of the Project; this explains the 25th month. Paper [18] answers (1) in negative by constructing a 3-generated lattice L such that L has no atom and so $|\mathrm{At}(L)| \notin \{1, 2, 3\}$. An atomless four-generated selfdual lattice is also constructed. Two of the three affirmative statements in [18] sound as follows. First, (1) holds for modular lattices. Second, if $|\mathrm{CoAt}(L)| = 0$, then $|\mathrm{At}(L)| \geq 1$. Paper [18] formulates the problem whether (1) is true with $\{0, 1, 2, 3\}$ instead of $\{1, 2, 3\}$. These results harmonize mostly with Section 6 "Study of special lattices" of the Research Plan.

References

All papers below indicate the support given from KH 126581
(Participants of the Project are typset in **boldface underlined**)

Written in months 1–12 of the project:

[1] ***OA*** {145} **Gábor Czédli**: A note on finite lattices with many congruences, *Acta Universitatis Matthiae Belii, Series Mathematics Online* (2018), 22-28;
`http://actamath.savbb.sk/oacta2018003.shtml`

[2] {147} Delbrin Ahmed, **Gábor Czédli**, and **Eszter K. Horváth**: Geometric constructibility of polygons lying on a circular arc. *Mediterranean J. of Math.* (2018) 15:13;
`https://rdcu.be/Q16Q` (upon subscription);
`http://arxiv.org/abs/1710.08859` (free preprint)

[3] {148} **Gábor Czédli**: Finite semilattices with many congruences. *Order* **36**, 233-247 (2019);
`https://rdcu.be/2Fzr` (upon subscription);
`https://arxiv.org/abs/1801.01482` (free preprint)

[4] ***OA*** {149} **Gábor Czédli** and Árpád Kurusa: A convex combinatorial property of compact sets in the plane and its roots in lattice theory. *Categories and General Algebraic Structures with Appl.* **11**, 57-92 (2019);
`http://cgasa.sbu.ac.ir/article_82639_995ede57b706f33c6488407d8fdd492d.pdf`

[5] ***OA*** {151} **Gábor Czédli**, **Gergő Gyenizse** and **Ádám Kunos**: Symmetric embeddings of free lattices into each other. *Algebra Universalis* (2019) 80:11;
`https://rdcu.be/bl0jN`

[6] ***OA*** {152} **Gábor Czédli**: Lattices with many congruences are planar. *Algebra Universalis* (2019) 80:16;
`https://rdcu.be/bpd58`

[7] ***OA*** {150} **Gábor Czédli**: Circles and crossing planar compact convex sets. *Acta Sci. Math. (Szeged)* **85** (2019), 337–353;
`https://doi.org/10.14232/actasm-018-522-2`

Written in months 13–24 of the project:

[8] ***OA*** {154} **Gábor Czédli**: Eighty-three sublattices and planarity. *Algebra Universalis* (2019) 80:45;
`https://rdcu.be/bVoPh`

[9] ***OA*** {156} **Gábor Czédli**: One hundred twenty-seven subsemilattices and planarity, *Order*,
`https://doi.org/10.1007/s11083-019-09519-x`

[10] {*} **Gábor Czédli**: Planar semilattices and nearlattices with eighty-three subnearlattices, *Acta Sci. Math. (Szeged)*, submitted;
`http://arxiv.org/pdf/1908.08155` (free preprint)

[11] {*} **Gábor Czédli**, Robert C. Powers, and Jeremy M. White: Medians are below joins in semimodular lattices of breadth 2. *Order*, submitted;
`http://arxiv.org/pdf/1911.02124` (free preprint)

[12] ***OA*** {153} **Gábor Czédli** and **Eszter K. Horváth**: A note on lattices with many sublattices. *Miskolc Math. Notes* **20** (2019) 839-848;
`http://mat76.mat.uni-miskolc.hu/mnotes/download_article/2821.pdf`

[13] **Eszter K. Horváth**, Sándor Radeleczki, Branimir Seselja, and Andreja Tepavcevic: Cuts of poset-valued functions in the framework of residuated maps. *Fuzzy Sets and Systems*, accepted,
`https://www.researchgate.net/publication/338533825 (free preprint)`

[14] ***OA*** {14} Delbrin Ahmed and **Eszter K. Horváth**: Yet two additional large numbers of subuniverses of finite lattices, *Discussiones Mathematicae General Algebra and Applications* **39** (2019) 251-261;
`https://content.sciendo.com/downloadpdf/journals/dmgaa/39/2/article-p251.xml`

[15] Endre Tóth and **Tamás Waldhauser**: Solution sets of systems of equations over finite lattices and semilattices. *Algebra Universalis*, accepted;
`https://arxiv.org/pdf/2001.04549` (free preprint)

[16] Kamilla Kátai-Urbán and **Tamás Waldhauser**: Multiplication of matrices over lattices. *Journal of Multiple-Valued Logic and Soft Computing*, submitted;
`https://arxiv.org/abs/2001.04656` (free preprint)

[17] Endre Tóth and **Tamás Waldhauser**: On centralizers of finite lattices and semilattices. *Journal of Multiple-Valued Logic and Soft Computing*, submitted;
`https://arxiv.org/abs/2001.04661` (free preprint)

Written in month 25 (the last month) of the project:

[18] {*} **Gábor Czédli**: On the number of atoms in lattices generated by few elements. *Acta Sci. Math. (Szeged)*, submitted;
`https://arxiv.org/pdf/2001.03188` (free preprint)

Gábor Czédli, principal investigator