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Dimension theory of non-conformal systems

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In this document I describe my research activities as the postdoctoral fellow of the grant NKFI PD123970 at the Department of Stochastics of the Mathematics Institute of Budapest University of Technology and Economics (BUTE) from 2017.09.01 to 2020.08.31.

During this period we have published 12 publications with my co-authors. Out of them, 3 have been already published in journals *Invent. Math., Proc. London Math. Soc., Ann. Acad. Sci. Fenn. Math.*; 4 have been accepted for publication or have appeared online in journals *Trans. Amer. Math. Soc., Israel J. Math., Int. Math. Res. Not., Math. Nachr.*; 2 conference proceedings have been accepted for publication in the proceedings of the conference "International Workshop and Conference on Topology & Applications", 5th December 2018 to 11th December 2018, Kochi, India; and 3 manuscripts.

During this period, my research interest focused on three main topics, the dimension theory of non-conformal systems; shrinking target problems and self-conformal systems.

I. Dimension theory of non-conformal sets and measures

Let $\Phi = \{f_i(x) = A_i x + t_i\}_{i=1}^N$ be a finite set of contracting invertible affine maps, mapping $\mathbb{R}^d$ into itself, that is, $\|A_i\| < 1$ for every $i = 1, \dots, N$. We call Φ self-affine iterated function system (IFS). Then there exists a unique non-empty compact set X, which is invariant with respect of the IFS Φ, i.e. $X = \bigcup_{i=1}^N f_i(X)$. We call the set X self-affine set or the attractor of Φ. Moreover, for every probability vector $\underline{p} = (p_1, \dots, p_N)$ there exists a unique compactly supported probability measure μ such that $\mu = \sum_{i=1}^N p_i (f_i)_* \mu$, where $f_* \mu = \mu \circ f^{-1}$ is the push forward of μ with respect to f. We call the measure μ self-affine measure or invariant measure of Φ. One of the cardinal questions in fractal geometry is to understand the structure of these geometric object, describe their regularity properties, and measure how large these objects are in the sense of measure and dimension.

Hutchinson [26] studied the special case of similarity maps, that is, $A_i = \lambda_i O_i$, where $0 < \lambda_i < 1$ and O_i is an orthonormal matrix. Under the open set condition (OSC), when there exists an open set U such that $\bigcup_i f_i(U) \subset U$ and $f_i(U) \cap f_j(U) = \emptyset$ for $i \neq j$, Hutchinson showed that the Hausdorff dimension of X equals to the similarity dimension, the unique solution of the equation $\sum_{i=1}^N \lambda_i^s = 1$. Moreover, he proved that μ is exact dimensional with dimension $\frac{\sum_{i=1}^N p_i \log p_i}{\sum_{i=1}^N p_i \log \lambda_i}$, which quantity is called Lyapunov dimension.

Falconer [17] generalized the similarity dimension to the more general self-affine case. In this case, the quantity is called affinity dimension. He showed that the affinity dimension is always an upper bound for the box counting dimension of X. Moreover, Solomyak [31] showed if $\|A_i\| < 1/2$ for every $i = 1, \dots, N$ then for Lebesgue typical choice of $(t_1, \dots, t_N)$ the Hausdorff dimension of X equals to the affinity dimension. Similar result was proven for the self-affine measures by Jordan, Pollicott and Simon [27] under the same condition. However, it was unknown for decades whether Hutchinson's Theorem can be generalized for self-affine sets, namely, whether it is possible to calculate the dimension of self-affine sets and measures under a kind of separation assumption.

We managed to solve this problem on $\mathbb{R}^2$ in Bárány, Hochman and Rapaport [4]. It was clear that Hutchinson's Theorem cannot hold in full generality. For instance, McMullen [30] showed a family of self-affine sets, which satisfies the strong separation condition (i.e. $f_i(X) \cap f_j(X) = \emptyset$ for $i \neq j$) and the matrices A_i are diagonal, and for which the Hausdorff and the box counting dimension are strictly smaller than the affinity dimension. Also an irreducibility condition on the matrices and the open set condition together is not enough. For example, in case of planar IFS $\Phi = f_1(x) = A_1 x, f_2(x) = A_2 x$, the matrices A_1

and A_2 with strictly positive entries can be chosen such that the matrices does not preserve any subspace, $f_i((0,1)^2) \subset (0,1)^2$ and $f_1((0,1)^2) \cap f_2((0,1)^2) = \emptyset$, however, the attractor X of Φ is a single point.

In Bárány, Hochman and Rapaport [4], we showed the following: if the planar self-affine IFS $\Phi = \{f_i(x) = A_i x + t_i\}_{i=1}^N$ satisfies

- the strong open set condition, that is, there is an open set $U \subset \mathbb{R}^2$ such that $\bigcup_i f_i(U) \subset U$, $f_i(U) \cap f_j(U) = \emptyset$ for $i \neq j$ and $X \cap U \neq \emptyset$;
- the matrices $\mathcal{A} = \{A_1, \ldots, A_N\}$ are strongly irreducible, i.e. there is no finite collection of subspaces $V_1, \ldots, V_N \subset \mathbb{R}^2$ which is invariant, that is $A_j(\bigcup_i V_i) = \bigcup_i V_i$ for every j

then the Hausdorff dimension of the self-affine set X equals to the singularity dimension and the dimension of the self-affine measure μ equals to the Lyapunov dimension. The proof relies on two tools. One is the Ledrappier-Young formula, which was proven for self-affine systems first by Bárány and Käenmäki [6], and which allows us to determine the dimension of the measures via studying the dimension of its orthogonal projection in the Furstenberg directions, i.e. the strongly contracting directions. The other tool was Hochman's Entropy Growth Theorem [22] and its adaptation for the projections of self-affine measures. In addition, Bárány, Hochman and Rapaport [4] showed that the dimension of projections of self-affine measures under the conditions above does not drop in any direction, that is, it takes a constant value in every direction, which is the largest possible value.

We provided two applications for the results of Bárány, Hochman and Rapaport [4]. Our first application was motivated by the question of Barnsley on fractal image compression. In Bárány, Rams and Simon [14], we studied the dimension theory of the more general family of dynamically defined fractal graphs, where the graph is the union of affine images of certain subsets of the graph. If the graph was self-affine, the graph would be the union of the images of the whole graph. This cause significant difficulties in the understanding of the underlying dynamics. For instance, the corresponding dynamics does not admit a Markov partition, hence, we needed to admit the techniques introduced by Hofbauer [24] and Hofbauer, Raith and Simon [25] to tackle the problems caused by the non-Markovian structure together with the results of Bárány, Hochman and Rapaport [4].

We (Bárány, Rams and Simon) also published a series of two conference proceedings on the dimension theory of some dynamically defined function graphs, like generalized Takagi and Weierstrass functions. In the first part [12] of the series, we presented a survey on the tools in the one-dimensional dynamics and non-conformal fractal theory that are needed to investigate the dimension theory of repellers of piecewise affine systems. In the second part [13], we studied the dimension of Markovian fractal interpolation functions and the generalised Takagi functions generated by non-Markovian dynamics.

We provided another application of Bárány, Hochman and Rapaport [4] in the theory of multifractal analysis. In Bárány, Jordan, Käenmäki and Rams [5], working on strongly irreducible planar self-affine sets satisfying the strong open set condition, we calculated the Birkhoff spectrum of continuous potentials and the Lyapunov spectrum. In more details, let $\Sigma = \{1, \ldots, N\}^{\mathbb{N}}$ be the symbolic space equipped with a natural metric. There exists a natural projection from Σ to X defined by $\pi(\mathbf{i}) = \lim_{n\to\infty} f_{i_1} \circ \cdots \circ f_{i_n}(0)$. Let $\varphi\colon \Sigma \mapsto \mathbb{R}^d$ be a continuous potential. In [5], for every $\beta \in \mathbb{R}^d$ we determined the Hausdorff dimension of the set of points $\pi(\mathbf{i}) \in X \subset \mathbb{R}^2$ for which the Birkhoff average $\frac{1}{n}\sum_{k=0}^{n-1} \varphi(\sigma^k \mathbf{i})$ converges to β, where σ is the left-shift operator on Σ. To do so, using thermodynamic formalism we approximated the dimension by n-step self-affine measures (i.e. which are self-affine with respect to the nth iteration of the IFS) for which the integral of $\varphi \circ \pi$ equals to the pre-given value β.

Moreover, for every $(\beta_1, \beta_2) \in \mathbb{R}^2$ we calculated also the Hausdorff dimension of the set of points $\pi(\mathbf{i}) \in X \subset \mathbb{R}^2$ for which the Lyapunov exponent $\chi_i(\mathbf{i}) = \lim_{n\to\infty} \frac{-1}{n} \log \alpha_i(A_{i_1} \cdots A_{i_n}) = \beta_i$, where $\alpha_i(A)$ is the ith singular value of A. In the d-dimensional non-conformal situation, the Lyapunov exponents are in general not given by a Birkhoff average of some potential. However, under the dominated splitting conditions, it is still true. The key idea was to calculate the Lyapunov spectrum on a sufficiently large subfamilies of the affine iterated function system which satisfy the dominated splitting condition by applying the result on Birkhoff averages.

This motivated our next problem. Recall that a cocycle of matrices $\mathcal{A} = \{A_1, \ldots, A_N\}$ is dominated if and only if there is a uniform exponential gap between singular values of its iterates. This is equivalent to the existence of a strongly invariant multicone in the projective space, see Avila, Bochi and Yoccoz [3]. An important corollary of this in the point of view of thermodynamic formalism that the semigroup $\mathcal{S}(\mathcal{A})$ generated by $\mathcal{A}$ is almost multiplicative, i.e. there exists a constant $\kappa > 0$ such that for every $A, B \in \mathcal{S}(\mathcal{A})$, $\|AB\| \geq \kappa\|A\|\|B\|$. In Bárány, Käenmäki and Morris [8], we showed in particular that for a planar matrix cocycle $\mathcal{A} = \{A_1, \ldots, A_N\}$ is almost multiplicative if and only if it is either conjugate to isometries, or satisfies a property slightly weaker than domination which is introduced in [8]. Moreover, we showed that for planar matrices, $\mathcal{A} = \{A_1, \ldots, A_N\}$ is dominated if and only if all the matrices are proximal and its almost multiplicative.

Finally, in the topic of the dimension theory of self-affine systems, we studied the Assouad dimension of planar self-affine sets in Bárány, Käenmäki and Rossi [9]. In the last few years, the concept of the Assouad dimension came to the forefront of the attention in the field of fractal geometry, see Fraser [19]. In [9], we determined the Assouad dimension of planar self-affine sets under the assumptions of strong irreducibility, strong open set condition and projection condition, which claims that the projection of the set is a line segment in certain directions.

II. Dimension and measure of shrinking target sets

The shrinking target problem is a general name for a class of problems, first investigated by Hill and Velani [20]. The setting is in the case of iterated function systems is as follows: Let $\Phi = \{f_i\}_{i=1}^N$ be an iterated function system of contracting $C^{1+\alpha}$ maps on $\mathbb{R}^d$ with attractor X, and let $\pi\colon \Sigma \mapsto X$ be the natural projection, moreover, let $\{B_k\}$ be a sequence of subsets of X. We call the set $\Gamma = \{\pi(\mathbf{i}) : \pi(\sigma^k\mathbf{i}) \in B_k \text{ for inf. many } k\text{'s}\}$ the shrinking target set, and the sets $\{B_k\}$ the targets.

In Bárány and Rams [11], we determined the Hausdorff dimension of the shrinking target sets for the Bedford-McMullen carpets, with targets being either cylinders of the form $\{f_{\mathbf{i}}(X)\}$ or geometric balls. This can be considered as the generalisation of Hill and Velani [21] on the plane.

If the IFS Φ consists of conformal mappings in $\mathbb{R}^d$, that is, $f_i'(x) = \lambda_i(x)O_i(x) \in C^\alpha$, where $0 < \lambda_i(x) < 1$ and $O_i(x)$ is an orthogonal matrix for every x in some open neighbourhood of X, then the Hausdorff dimension of shrinking target sets has been determined by Hill and Velani [20] in the case when the targets are geometric balls and open set condition holds. However, the value of the Hausdorff measure at the Hausdorff dimension was unknown. Levesley, Salp and Velani [29] showed that the Hausdorff measure of the shrinking target set in the middle-third Cantor satisfies a zero-full dichotomy. That is, it is either zero or full according to the convergence or divergence of a certain sum which is dependent on the radii of the targets. In Allen and Bárány [1], we proved an analogue of this result, obtaining a zero-full dichotomy for the Hausdorff measure of shrinking targets, in the setting of more general self-conformal sets with OSC. Unlike in the work of Levesley, Salp, and Velani [29], we showed that the Mass Transference Principle due to Beresnevich and Velani [16] is unable to be applied in this setting. Instead, our proof relies on recasting the problem in the language of symbolic dynamics and appealing to several concepts from thermodynamic formalism, eventually enabling us to use an analogue of the mass distribution principle.

III. Geometric measure theory of self-conformal sets

As we have seen, the dimension of self-similar sets and measures is well understood under certain separation conditions (like OSC), see Hutchinson [26]. The situation becomes more difficult when overlaps occur in the structure of the self-similar set. In his seminal paper, Hochman [22] showed that exponential separation between the cylinders suffices for the equality of the Hausdorff and similarity dimension. In particular, if the Hausdorff dimension is strictly smaller than the similarity dimension then distance $\min_{\mathbf{i}\neq\mathbf{j}\in\{1,\ldots,N\}^n} \|f_{\mathbf{i}}(0) - f_{\mathbf{j}}(0)\|$ decays superexponentially, where $f_{\mathbf{i}} = f_{i_1} \circ \cdots f_{i_n}$ for $\mathbf{i} = (i_1, \ldots, i_n)$.

A folklore conjecture proposes that the only possibility for the Hausdorff dimension to be strictly less than the similarity dimension is the existence of exact overlaps, i.e. there exist $\mathbf{i} \neq \mathbf{j} \in \Sigma^*$ such that $f_{\mathbf{i}} \equiv f_{\mathbf{j}}$. Hochman asked in [23] if the super-exponential condensation further implies the exact overlapping. In Bárány and Käenmäki [7], we answered this question negatively by constructing a parametrized family of self-similar sets for which there are uncountably many parameters such that the system has super-exponential condensation but no exact overlaps.

In Bárány and Szvák [15], we studied the dimension of self-similar measures on the line for which Hochman's result [22] is not applicable. We determined the Hausdorff dimension of the self-similar measures of the IFS $\{\alpha x, \beta x, \gamma x + (1-\gamma)\}$. Since the first two maps share the same fixed point, there are exponentially many exact overlaps, moreover, the weak separation property does not hold if $\log \alpha / \log \beta \notin \mathbb{Q}$. We provided an "almost every" type result if $0 < \alpha, \beta, \gamma < 1/9$ by a direct application of the results of Feng and Hu [18] and Kamalutdinov and Tetenov [28].

Finally, in Bárány, Kolossváry, Rams and Simon [10], we considered self-conformal IFSs on the real line where there exist overlaps between the first level cylinders. In the space of self-conformal IFSs, we showed that generically (in topological sense) if the attractor of such a system has Hausdorff dimension less than 1 then it has zero appropriate dimensional Hausdorff measure and its Assouad dimension is equal to 1. Our main contribution was in showing that if the cylinders intersect then the IFS generically does not satisfy the weak separation property and hence, we may apply a recent result of Angelevska, Käenmäki and Troscheit [2].

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