ANALYSIS AND APPLICATION OF INNOVATIVE INTEGRATORS

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Numerical solution of ordinary and partial differential equations is inevitable when forecasting the time-behaviour of natural, financial, or even social scientific phenomena. Since these processes are quite complicated, an exact solution to their mathematical model is usually unknown. The numerical treatment means the approximation of the exact solution. The question arises how "good" the numerical method is. Its usefulness can be described by certain notions and quantities studied in the framework of numerical analysis. The possibly most important property is the convergence which ensures the approximate solution to get closer to the exact one when refining the parameters of the numerical method. There are other issues to be investigated too, e.g., stability, conservation of qualitative properties, order of the convergence, etc. When introducing a new numerical method for solving the mathematical model, it is always necessary to analyse the properties listed above.

The main aim of this project was to investigate the convergence, stability, and qualitative properties of innovative time integrators (operator splitting methods, exponential integrators, and Magnus integrators). We also aimed at implementing the methods by computer.

To do so we considered a whole class of problems by presenting them in an abstract form of evolution equations. This made us possible to treat them in an abstract framework by using functional analytic tools, especially operator semigroup theory.

In the present report we summarise the results obtained in the framework of the project. We set first the problem and introduce the innovative integrators considered. Then our results are presented according to the publications they appear in.

1 PROBLEM SETTING

To treat all time-dependent partial differential equations belonging to various classes in the same framework, we consider a Banach space X endowed with the norm $\|\cdot\|$, and a nonlinear and unbounded operator $\mathscr{A}$ mapping from a subset $D(\mathscr{A})$ of X to X. We consider the following nonlinear abstract Cauchy problem corresponding to the operator $\mathscr{A}$ on X for the function $u\colon [0,+\infty) \to X$:

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}u(t) = \mathscr{A}(u(t)), \;\; t>0, \\ \;\; u(0)=u_0 \in X \;\text{ given.} \end{cases} \tag{NACP}$$

We list here the form of operator $\mathscr{A}$ appearing in the problems we treated in the project. To operator semigroup theory our main reference is Engel–Nagel [4].

(a) **Infinite dimensional linear regulator problems** are described by the operator

$$\mathscr{A}(u(t)) = (A+WR)u(t) \tag{LQR}$$

where A is a linear operator which generates a strongly continuous semigroup on X, W is a certain weight operator, and operator R is the solution to the algebraic operator Riccati equation.

(b) **Linearised shallow water equations** in $X = (\mathrm{L}^2(\Omega))^3$, $\Omega \subset \mathbb{R}^2$, involve the linear operator

$$\mathscr{A}(u(t)) \equiv Au(t) \quad \text{with}$$
$$A = -\begin{pmatrix} \overline{u}\partial_x + \overline{v}\partial_y + \partial_x\overline{u} + \partial_y\overline{v} & \overline{h}\partial_x + \partial_x\overline{h} & \overline{h}\partial_y + \partial_y\overline{h} \\ c_{\mathrm{F}}\partial_x & \overline{u}\partial_x + \overline{v}\partial_y + \partial_x\overline{u} & -c_{\mathrm{R}} + \partial_y\overline{u} \\ c_{\mathrm{F}}\partial_y & c_{\mathrm{R}} + \partial_x\overline{v} & \overline{u}\partial_x + \overline{v}\partial_y + \partial_y\overline{v} \end{pmatrix} \tag{LWS}$$

and the domain

$$D(A) = \big\{(u_1, u_2, u_3) \in \big(\mathrm{L}^2(\Omega)\big)^3 \colon u_1 \in \mathrm{H}^1(\Omega),\ \operatorname{div}\tbinom{u_2}{u_3} \in \mathrm{L}^2(\Omega),\ \big\langle\tbinom{u_2}{u_3}, \mu\big\rangle_{\mathbb{R}^2} = 0\big\},$$

where $\langle\cdot,\cdot\rangle_{\mathbb{R}^2}$ denotes the inner product in $\mathbb{R}^2$, $\mu \in \mathbb{R}^2$ is the normal vector field of the boundary $\partial\Omega$, and $c_{\mathrm{F}}, c_{\mathrm{R}} > 0$ are physical constants.

(c) **Dissipative delay equations** are given by the operator

$$\mathscr{A}(u(t)) = Au(t) + \Phi u_t \tag{DDE}$$

where A is a linear operator which generates a strongly continuous semigroup on X being a Hilbert space in this case, $\Phi\colon \mathrm{L}^p([-1,0], X) \to X$, and the history function $u_t\colon [-1,0] \to X$ is defined by $u_t(s) = u(t+s)$ for all $s \in [-1,0]$ and $t \geq 0$.

(d) **Quasilinear delay equations** involve the operator

$$\mathscr{A}(u(t)) = Q(u(t-\delta))u(t) \tag{SDE}$$

with $\delta > 0$ and an operator $Q\colon X \to \mathscr{L}(X)$ describing the underlying processes.

(e) **Linear switched systems** involve an operator that is piecewise constant in time. The switch between the constant linear operators A_j, $j = 1, \ldots, J$, $J \in \mathbb{N}$, is given by the switch rule $\sigma\colon [0,\infty) \to \{1, \ldots, J\}$ which assign an index to all time values $t > 0$. Hence, the operator in (NACP) has the following form:

$$\mathscr{A}(u(t)) = A_{\sigma(t)}u(t) \quad \text{with} \quad A_{\sigma(t)} = A_j \quad \text{for} \quad t \in (t_\ell, t_{\ell+1}], \tag{LSS}$$

where t_ℓ, $\ell = 1, \ldots, \ell_{\max}$ denote the discontinuity points of σ.

2 INNOVATIVE INTEGRATORS

In what follows we introduce three problem classes and the corresponding innovative integrators.

Our main goal is to approximate the solution u to problem (NACP) at certain time levels t_n, $n \in \mathbb{N}$. To this end we define a time step $\tau > 0$ and denote by $u_n \in X$ the approximation to $u(t_n)$ at time levels $t_n = n\tau$ for all $n \in \mathbb{N}$. For the sake of simplicity we take u_0 to be the initial value in (NACP).

(i) **Operator splitting procedures** are efficient tools to solve problem (NACP) if the operator $\mathscr{A}$ is the sum of $\mathscr{A}_j$ for $j = 1, ..., j_{\max}$, $j_{max} \in \mathbb{N}$. By assuming that each of them generates a strongly continuous nonlinear semigroup

$$\mathcal{S}_j(t) = \lim_{k\to\infty} \big((\mathrm{I} - \tfrac{t}{k}\mathscr{A}_j)^{-1}\big)^k,$$

the approximate solution $u_n \approx u(n\tau)$ to (NACP) is given e.g. by the sequential splitting as

$$u_n = \big(\mathcal{S}_1(\tau)\mathcal{S}_2(\tau)\ldots\mathcal{S}_j(\tau)\big)^n(u_0) \quad \text{for} \quad n \in \mathbb{N},$$

where the power denotes the n times iteration of the corresponding operator.

(ii) **Exponential integrators** were derived for solving semilinear problems efficiently, that is, when $\mathscr{A}(u(t)) = Lu(t) + g(u(t))$, where L is a linear operator generating the stongly continuous semigroup $(\mathrm{e}^{\tau L})_{\tau \geq 0}$ and $g\colon X \to X$ is a nonlinear function with some "nice" properties. The simplest one is the exponential Euler method:

$$u_n = \mathrm{e}^{\tau L} u_{n-1} + \int_0^\tau \mathrm{e}^{(\tau - s)L} g(u_{n-1})\,\mathrm{d}s \quad \text{for all} \quad n \in \mathbb{N}.$$

(iii) **Magnus integrators** were originally developed for linear non-autonomous problems in Magnus [8]. The solution to special nonlinear problems, the quasilinear problems with $\mathscr{A}(u(t)) = Q(u(t))u(t)$ in (NACP) for some given linear operator $Q\colon D(Q) \subset X \to X$, can be approximated by constructing and solving a sequence of appropriate linear non-autonomous problems by Magnus integrators. An example for Magnus integrator for quasilinear problems reads as

$$u_n = \mathrm{e}^{\tau Q(u_{n-1/2})} u_{n-1} \quad \text{with} \quad u_{n-1/2} = \mathrm{e}^{\frac{\tau}{2} Q(u_{n-1})} u_{n-1} \quad \text{for all} \quad n \in \mathbb{N}, \tag{1}$$

where the exponential notation refers to the semigroup generated by the corresponding operator.

In what follows we summarise our scientific results obtained during the project.

3 Well-Posedness of Linearised Shallow Water Equations

The present section is based on the following publication:

> P. Csomós, J. Winckler: A semigroup proof for the well-posedness of the linearised shallow water equations, *Analysis Mathematica* **43** (2017) 445–459.
> `https://doi.org/10.1007/s10476-017-0204-7`

Shallow water equations represent a special case of Navier–Stokes equations for incompressible and inviscid fluids moving on the rotating planet. They played an important role in the first attempts to describe the atmosphere's large-scale dynamics. Numerical weather prediction models and ocean dynamical models, e.g. tsunami forecasting, are still based on them.

Instead of only referring to the shallow water equations as presented in the literature, we gave an insight into their derivation. We then obtained the linearised shallow water equations (LWS), which is needed frequently when solving the problem numerically (cf. exponential integrators). In order to show that the linearised version is well-posed, we rewrote it as an abstract Cauchy problem (LWS) on the Hilbert space $X = (\mathrm{L}^2(\Omega))^3$, $\Omega \subset \mathbb{R}^2$. Then Stone's theorem yielded that the corresponding operator A generated a strongly continuous unitary group, hence, the problem was well-posed. More precisely, we proved the following result with the help of Kurula–Zwart [7].

Proposition 3.1. *Suppose that $(\overline{h}, \overline{u}, \overline{v}) \in \mathbb{R}^3$ satisfies the following three conditions:*

$$\langle \tbinom{\overline{u}}{\overline{v}}, \mu \rangle_{\mathbb{R}^2} = 0 \ \text{on}\ \Omega \subset \mathbb{R}^2, \ \operatorname{div}\left(\overline{h}\tbinom{\overline{u}}{\overline{v}}\right) = 0, \ c_F \operatorname{grad} \overline{h} - c_R \begin{pmatrix} \overline{v} \\ -\overline{u} \end{pmatrix} + \begin{pmatrix} \langle \tbinom{\overline{u}}{\overline{v}}, \operatorname{grad} \overline{u} \rangle_{\mathbb{R}^2} \\ \langle \tbinom{\overline{u}}{\overline{v}}, \operatorname{grad} \overline{v} \rangle_{\mathbb{R}^2} \end{pmatrix} = 0.$$

Then the operator A with domain $D(A)$ defined in (LWS)*, respectively, is the generator of a unitary group with respect to $\langle \cdot, \cdot \rangle_X$ if and only if $\overline{u} = \overline{v} = 0$ and $\overline{h} \in (h_{min}, h_{max}) \subset \mathbb{R}$ is constant.*

4 Operator Splitting for Solving Linear Quadratic Regulator Problems

The present section is based on the following publication:

> P. Csomós, H. Mena: Fourier-Splitting method for solving hyperbolic LQR problems, *Numerical Algebra, Control and Optimization* **8** (2018) 17–46.
> http://dx.doi.org/10.3934/naco.2018002

Optimal control problems are used to determine the suitable way how to drive a system into its predefined state meanwhile a cost functional is minimised. We treated the linear quadratic regulator problem (LQR) where operator A refers to the dynamic, and operator $C := WR$ describes the control.

The use of operator splitting made it possible to approximate the optimal state $u(t) = \mathrm{e}^{t(A+C)}u_0$ by the composition of the separately computed semigroups $(\mathrm{e}^{\tau A})_{\tau\geq 0}$ and $(\mathrm{e}^{\tau C})_{\tau\geq 0}$. In many applications, the operator $\mathrm{e}^{\tau A}$ can be almost accurately obtained by using Fourier transform. Moreover, in our approach the operator $\mathrm{e}^{\tau C}$ should be computed and stored only once at the very beginning of the computation.

As an application, we treated the one- and two-dimensional wave equations as well as the one- and two-dimensional linearised shallow water equations (LWS). Our goal was to avoid flood on the shore, therefore, the system should be driven to the zero state. We treated distributed control (acting at all points of the domain) and another one imitating a sink.

In this case X denoted a Hilbert space and we considered problem (NACP) with operator (LQR). For $N \in \mathbb{N}$, we introduced the projection $P_N \colon X \to \mathbb{R}^N$ and the interpolation $J_N \colon \mathbb{R}^N \to X$ with the properties

$$P_N J_N = \mathrm{I}_N, \text{ the identity operator in } \mathbb{R}^N, \text{ and } \lim_{N\to\infty} J_N P_N x = x \text{ for all } x \in X,$$

being natural requirements in the framework of spatial discretisations. We got then the following result.

Proposition 4.1. *Let $(A, D(A))$ be a generator and $C \in \mathscr{L}(X)$. Then the combined numerical methods are convergent, that is, for all $u_0 \in X$ initial value, we have the following limits:*

$$u(t) = \lim_{N,n\to\infty} J_N \big(F_N^{-1}\mathrm{e}^{\frac{t}{n}\widehat{A}}F_N \mathrm{e}^{\frac{t}{n}C}\big)^n P_N u_0 \qquad \textit{(sequential splitting)},$$

$$u(t) = \lim_{N,n\to\infty} J_N \big(F_N^{-1}\mathrm{e}^{\frac{t}{2n}\widehat{A}}F_N \mathrm{e}^{\frac{t}{n}C}F_N^{-1}\mathrm{e}^{\frac{t}{2n}\widehat{A}}F_N\big)^n P_N u_0 \qquad \textit{(Strang–Marchuk splitting)}$$

with discrete Fourier transform $F_N \in \mathscr{L}(X)$, where the convergence is uniform for t in compact intervals.

We included the numerical scheme's algorithm, and added numerical experiments illustrating our results.

5 Operator Splitting for Dissipative Delay Equations

The present section is based on the following publication:

> A. Bátkai, P. Csomós, B. Farkas: Operator splitting for dissipative delay equations, *Semigroup Forum* **95** (2017) 345–365.
> https://doi.org/10.1007/s00233-016-9812-y

We investigated the convergence and its order for the Lie–Trotter product formulae for delay equations (DDE) by using the theory of nonlinear contraction semigroups in Hilbert spaces. We had a linear abstract Cauchy problem on the space $\mathcal{E} = X \times \mathrm{L}^p\big([-1,0], X\big)$ with the operator written as the sum

$$\mathcal{G} = \begin{pmatrix} B & \Phi \\ 0 & \frac{\mathrm{d}}{\mathrm{d}s} \end{pmatrix} = \begin{pmatrix} B & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & \Phi \\ 0 & \frac{\mathrm{d}}{\mathrm{d}s} \end{pmatrix} = \mathcal{G}_1 + \mathcal{G}_2 \qquad (2)$$

where I is the identity operator on $\mathrm{L}^p([-1,0],X)$, with the corresponding domains in $\mathcal{E}$. We obtained the following results on the convergence and its order.

Proposition 5.1. *Under suitable assumptions (e.g. m-dissipativity), for every $p \in (1,\infty)$ the sequential splitting converges in $X \times \mathrm{L}^p([-1,0],X)$ in a properly chosen equivalent norm.*

Proposition 5.2. *Assume that A generates a linear contraction semigroup and that the linear operator Φ satisfies the condition $\operatorname{ran}\Phi \subset D(A)$. Then for initial values in $D(\mathcal{G}^2)$, the sequential splitting is convergent of first order.*

We had similar result for the Lie splitting which involves the resolvents of the sub-operators. We illustrated our results by numerical experiments.

6 MAGNUS-TYPE INTEGRATOR FOR QUASILINEAR DELAY EQUATIONS WITH AN APPLICATION TO EPIDEMIC MODELS

The present section is based on the following publication:

> P. Csomós: Magnus-type integrator for semilinear delay equations with an application to epidemic models, *Journal of Computational and Applied Mathematics* **363** (2020) 92–105.
> `https://doi.org/10.1016/j.cam.2019.05.031`

In this paper we applied the Magnus method (1), being originally developed for non-autonomous problems by Magnus in [8], to the quasilinear delay equation of form (SDE). We first proved its second-order convergence and its positivity preserving property.

Proposition 6.1. *Under certain assumptions on function Q, the Magnus integrator* (1) *is convergent of second-order when applied to the quasilinear delay equation* (SDE)*, that is, there exists a constant $C > 0$, independent of τ and n, such that the error estimate $\|u(t_n) - u_n\| \leq C\tau^2$ holds for all $n \in \mathbb{N}_0$ and $\tau \geq 0$ with $t_n = n\tau \in [0,T]$.*

Proposition 6.2. *Let $Q\colon \mathbb{R}^d \to \mathbb{R}^{d\times d}$ be a function such that $Q(w)$ is a Metzler matrix for all positive vectors $w \in \mathbb{R}^d$. Then the Magnus integrator* (1) *preserves the positivity, that is, u_{n+1} is a positive vector for positive vector u_n and positive initial history function.*

As an application we dealt with the delayed epidemic models. Let $S, I, R\colon \mathbb{R}_0^+ \to [0,1]$ denote the density of susceptible, infected, and recovered humans among the total population, respectively. For their temporal change, a compartment-type model can be formulated with infection rate $\beta > 0$, recovery rate $\gamma > 0$, and latent period $\delta > 0$. Then for all $t > 0$, we considered the delayed epidemic model (based on Kermack–McKendrick [6] and Cooke [3]):

$$S'(t) = -\beta S(t)\frac{I(t-\delta)}{1+\alpha I(t-\delta)}, \quad I'(t) = \beta S(t)\frac{I(t-\delta)}{1+\alpha I(t-\delta)} - \gamma I(t), \quad R'(t) = \gamma I(t) \tag{3}$$

with $\alpha \in \{0,1\}$, which could be written in form (SDE) fulfilling the assumptions of the propositions above.

7 OPERATOR SPLITTING FOR SPACE-DEPENDENT EPIDEMIC MODEL

The present section is based on the following publication:

> P. Csomós, B. Takács: Operator splitting for space-dependent epidemic model, submitted to *Applied Numerical Mathematics*
> http://real.mtak.hu/id/eprint/104553

We presented and analysed numerical methods with operator splitting procedures (sequential, weighted sequential, and Strang–Marchuk), applied to an epidemic model which is similar to (3) but takes into account the space-dependence of the infection, too. We derived conditions on the time step, under which the numerical methods preserve the positivity and monotonicity properties of the exact solution. We were also after the cases when our method gave higher bound on the time step than the one already presented in the literature. We illustrated our theoretical results by numerical experiments.

Proposition 7.1. *(a) The total size of the population is preserved without any restriction.*

(b) Properties $S_{n+1} \geq S_n$ and $R_{n+1} \leq R_n$ are consequences of the positivity preservation, that is, $X_n \geq 0$ implies $X_{n+1} \geq 0$ for all $n \in \mathbb{N}$ and $X \in \{S, I, R\}$.

(c) The positivity preservation holds for certain $\tau \in \mathbb{T} \subset \mathbb{R}$ time steps, where the exact form of set $\mathbb{T}$ varies for the various splitting schemes and is presented in the paper.

We showed that in case of "rapid" recovery (i.e., γ is grater then a critical value), the sequential splitting needed no restriction on the time step to yield non-negative population densities. Hence, it behaves qualitatively better than the method which does not use operator splitting. Furthermore, sequential splitting requires time step from a broader interval as the method without splitting also for "slow" recovery. The same behaviour was observed in the case of weighted and Strang–Marchuk splittings, too. Namely, we obtained a larger upper bound for the time step than the reference one.

8 MAGNUS METHOD FOR QUASILINEAR EVOLUTION EQUATIONS WITH DELAY

> This is an ongoing work together with Dávid Kunszenti-Kovács with the working title *Magnus method for quasilinear evolution equations with delay*.

Our aim is to generalise the results of Section 6, that is, to prove the second-order convergence of Magnus integrator when applied to quasilinear delay equations (SDE), in the case when operator $Q(\cdot)$ maps from a Banach space X to X, and is not necessarily bounded.

Proposition 8.1. *Under certain assumptions on the function $Q(\cdot)\colon X \to X$, the Magnus method* (1) *is convergent of second order when applied to the quasilinear delay equation* (SDE)*, i.e., there exists constant $C > 0$, being independent of n and τ , such that the global error estimate $\|u(t_n) - u_n\| \leq C\tau^2$ holds for all $n \in \mathbb{N}$.*

In the proof, we first reformulated the difference between the exact solution $u(t_n)$ and u_n by the triangular inequality and got

$$\|u(t_n) - u_n\| \leq \|u(t_n) - \widehat{u}_n\| + \|\widehat{u}_n - u_n\| \tag{4}$$

where $\widehat{u}_n$ denotes the numerical solution in González et al. [5]. Thus, the first term could be estimated by their result. To this end we needed to ensure that the terms

$$\begin{aligned} g_n(t) &= \big(A(t) - A(t_n + \tfrac{\tau}{2})\big)u(t), \quad t \in [t_n, t_{n+1}], \\ \|g_n\|_{X,\infty} &= \max\{\|g_n(t)\|_X \colon t \in [t_n, t_{n+1}]\}, \\ \|g\|_{X,\infty} &= \max\{\|g_n\|_{X,\infty} \colon n \in \mathbb{N}, t_{n+1} \in [0, T]\}, \end{aligned}$$

appearing in the convergence result in González et al. [5], are well-defined. When bounding the second term in (4) by the telescopic identity, we arrived at a point when the norm convergence of the Magnus method for non-autonomous problems was needed. Here we referred to the results of Bátkai–Sikolya [2]. Then a version of the discrete Gronwall lemma yielded the second-order convergence. We also formulated the conditions on Q under which the Magnus method (1) preserved the non-negativity when applied to (SDE). Here we used the characterisation of positive operators from Bátkai et al. [1].

9 MATHEMATICAL ANALYSIS OF A SWITCHED SYSTEM

This is an ongoing research, initiated by Zsolt Horváth, with the working title *Analysis of a switched system in diesel engine modelling*.

Switched systems are often used in engineering tasks. The time-dependent operator $A(t)\colon X \to X$ appearing in the non-autonomous Cauchy problem is then piecewise constant in t, see in (LSS). If the system has a given input function $w\colon [0,T] \to X$ or some type of feedback, it appears as an additive term in the equation. We examined the problem where the linear switched system was obtained by linearising a nonlinear problem around certain states u_k^*. Then the time-dependent linear operators are derived from the corresponding partial derivatives of the nonlinear function. Then the solution $\overline{u}_\ell\colon [0,T] \to X$ to the resulting problem of form

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}\overline{u}_\ell(t) = A_{\sigma(t_\ell^*)}\overline{u}_\ell(t) + B_{\sigma(t_\ell^*)}\overline{w}_\ell(t), \quad t \in (t_\ell, t_{\ell+1}] \\ \overline{u}_\ell(t_{\ell-1}) = \overline{u}_{\ell-1}(t_{\ell-1}) + u^*_{\sigma(t^*_{\ell-1})} - u^*_{\sigma(t^*_\ell)} \ \text{ with } \ t_\ell^* = \frac{1}{2}(t_\ell + t_{\ell+1}) \ \text{ for all } \ \ell = 1, \dots, L \end{cases} \tag{5}$$

approximates the difference between $u^*_{\sigma(t^*_\ell)}$ and the solution of the nonlinear equation. We already determined the exact form of the approximate solution.

Proposition 9.1. *The solution to problem* (5) *reads as*

$$\begin{aligned} \overline{u}_\ell(t) &= \mathrm{e}^{(t-t_\ell)A_{\sigma(t_\ell^*)}}\overline{u}_\ell(t_{\ell-1}) + \int_{t_\ell}^t \mathrm{e}^{(t-s)A_{\sigma(t_\ell^*)}} B_{\sigma(t_\ell^*)}\overline{w}_\ell(s)\,\mathrm{d}s \\ &= \mathrm{e}^{(t-t_\ell)A_{\sigma(t_\ell^*)}}\big(\overline{u}_{\ell-1}(t_{\ell-1}) + u^*_{\sigma(t^*_{\ell-1})} - u^*_{\sigma(t^*_\ell)}\big) + \int_{t_\ell}^t \mathrm{e}^{(t-s)A_{\sigma(t_\ell^*)}} B_{\sigma(t_\ell^*)}\overline{w}_\ell(s)\,\mathrm{d}s \end{aligned} \tag{6}$$

for all $t \in [t_\ell, t_{\ell+1}]$, $\ell = 0, \dots, L$ *with* $\overline{u}_0(t_0) = u_0 - u^*_{\sigma(t_0)}$.

Our next tasks are the following.

(i) We will prove of the convergence of $\overline{u}_\ell(t)$ to the exact solution of the original nonlinear problem.

(ii) We will determine the order of the above convergence. To achieve this goal, we treat the problem with the specific switching rule

$$\sigma(t) = \Big[K\frac{w(t)-a}{b-a}\Big] + 1 \ \text{ with } \ a = \min_{t\in[0,T]} w(t), \ b = \max_{t\in[0,T]} w(t), \ \text{ and } \ K \in \mathbb{N} \ \text{ parameter,}$$

which originates from an existing engineering problem. We guess that we should obtain a first-order convergence in the parameter K.

(iii) We plan to study the various stability concepts of switched systems, e.g. the bounded input – bounded output stability.

10 SCIENFIFIC ACTIVITIES

In this section I list my scientific activities which took place or started during the project.

Participation in research activities

- 10.2019–09.2023: Management Committee member, COST Action CA18232 *Mathematical models for interacting dynamics on networks*
- 10.2019–09.2023: Vice-Leader of Working Group *Numerical methods and applications*, COST Action CA18232 *Mathematical models for interacting dynamics on networks*
- 01.2019–12.2020: Hungarian Project Leader in bilateral co-operation entitled *Coupled Systems and Innovative Time Integrators* (308019) with Bergische Universität Wuppertal (Germany) between Tempus Public Foundation (Hungary) and the German Academic Exchange Service (DAAD)
- 09.2018–08.2021: Bolyai János Research Scholarship of the Hungarian Academy of Sciences

Conference attendance

- 17-19.06.2019: "Women in Operator Theory and its Applications", Lisbon, Portugal,
  talk on *Operator semigroups for numerical analysis* (invited talk)
- 27-29.05.2019: "2$^{\text{nd}}$ Bergen–Budapest Workshop on Qualitative and Numerical Aspects of Mathematical Modelling", Bergen, Norway,
  talk on *Operator Splitting: From Theory to Application* (invited talk)
- 29.04.2019: "Young researchers' mini-symposium", Alfréd Rényi Institute of Mathematics, Budapest,
  talk on *Innovative time integrators and their application*
- 18-22.02.2019: "90$^{\text{th}}$ GAMM Annual Meeting", Vienna, Austria,
  talk on *Efficient numerical methods for solving optimal control problems*
- 03-07.09.2018: "European Women in Mathematics General Meeting 2018", Graz, Austria,
  talk on *Operator semigroups for innovative time integrators* (invited talk)
- 18-22.06.2018: "The 20$^{e}xtth$ European Conference on Mathematics for Industry", Budapest, Hungary,
  talk on *Flood prevention with mathematics*
- 11-16.06.2018: "Seventh Conference on Finite Difference Methods: Theory and Applications", Lozenetz, Bulgaria,
  talk on *Innovative integrators for optimal control of shallow water equations*

Participation in workshop organisation

- 15-17.04.2018: mini-workshop on DAAD project *Coupled Systems and Innovative Time Integrators*, Eötvös Loránd University, Budapest
- 18-22.06.2018: minisymposium *Differential Equations in Numerical Modelling: From Theory to Application*, "The 20$^{e}xtth$ European Conference on Mathematics for Industry", Budapest, Hungary
- 11-16.06.2018: special session *Numerical Methods for Propagation Processes*, "Seventh Conference on Finite Difference Methods: Theory and Applications", Lozenetz, Bulgaria

Seminar talks

- 21.10.2019: *Analysis of a switched system*, Department of Applied Analysis and Computational Mathematics, Eötvös Loránd University, Budapest

- 06.05.2019: *Application of operator splitting methods in turbulent models*, Department of Applied Analysis and Computational Mathematics, Eötvös Loránd University, Budapest
- 19.11.2018: *Innovative integrators and the study of their convergence*, Department of Applied Analysis and Computational Mathematics, Eötvös Loránd University, Budapest
- 15.11.2018: *Operator semigroups, innovative integrators, flood protection*, Farkas Miklós Seminar, Budapest University of Technology and Economics
- 20.04.2017: *Innovative integrators*, Farkas Miklós Seminar, Budapest University of Technology and Economics

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