FINAL REPORT ON THE PROJECT SNN 116095

– Combinatorial problems with an emphasis on games –

MEMBERS OF THE PROJECT

Our research was carried out by five members of the project - Csilla Bujtás, György Dósa, Balázs Patkós, Zsolt Tuza, Máté Vizer - in active collaboration with colleagues from our Institute and also abroad.

COLLABORATION BETWEEN THE HUNGARIAN AND SLOVENIAN RESEARCH GROUPS

In the framework of this project, active scientific collaboration has been established between the two groups, visiting each other on a regular basis. Moreover, we organized a workshop in the subject of the project (Balatonalmádi 5–10 June 2017), with four keynote speakers and 21 further participants.

PUBLICATIONS

About the new scientific results achieved during the project, the members of our group wrote 72 papers. From them 48 already appeared in SCI journals with impact factor; and more than 15 are still in the refereeing process, from which several further SCI publications are expected. The complete list of published or submitted papers is given on a separate web page of the final report.

SCIENTIFIC RESULTS

Below we list a selection of results achieved during the project. The description does not intend to be complete, but we try to give at least a representative selection from the many problems studied.

DOMINATION: STATIC PROBLEMS AND GAME VERSIONS

In a graph G, a sequence v_1, v_2, …, v_s of vertices is Grundy dominating if every v_i dominates a so far undominated vertex of G. Depending on whether domination is considered according to closed or open neighborhood of v_i and the previous vertices, this vague definition can be interpreted in four ways. The original notion (closed neighborhoods vs closed neighborhoods) has been studied in some detail, the other three variants were introduced in our recent works. The relations of the other three are investigated, but perhaps most importantly the equivalence of one of the new variants to the zero forcing number of G is established. Further, on explicitly defined graph products, we determined the maximum length of dominating sequences for almost all possible 2-

term products of paths and/or cycles for all four types of products: Cartesian, lexicographic, direct, strong. Beyond very different proofs of upper bounds, our general observations can be useful when dealing with other classes of graphs, too.

The total domination number gamma_t(G) of a graph G is the order of a smallest vertex set D such that each vertex of the graph is adjacent to some vertex from D. The annihilation number a(G) of G is the largest integer k such that there exist k different vertices in G with degree sum of at most |E(G)|. It is conjectured that gamma_t(G) is at most a(G)+1 for every nontrivial connected graph. We proved that the conjecture is true for cactus graphs and block graphs.

In a game setting, the total domination game is a version of the domination games where the rules are given in terms of the open neighborhoods of the vertices instead of the closed ones. The related graph invariant is the game total domination number. We proved that the game total domination is at most 11n/14 for every n-vertex graph which does not contain isolated vertices or isolated edges. This improves the earlier best upper bound 4n/5, although it still leaves open the question whether the conjectured 3n/4 is a valid general upper bound.

Another variant of dominating sets is so-called 2-domination. It is a 1-fault-tolerant version of domination in graphs: a set D of vertices is 2-dominating if every vertex not in D has at least two neighbors in D. The minimum size of a 2-dominating set is the 2-domination number. We considered the 2-domination number of graphs with minimum degree at least 6. We established upper bounds on the 2-domination number, improving the best earlier bounds for any minimum degree between 6 and 21. In particular, we proved that the 2-domination number is strictly smaller than n/2 if the minimum degree of the graph is at least 6. Further we considered a large class of graphs and characterized those members which have equal domination and 2-domination numbers. For the general case, we proved that it is NP-hard to decide whether this equality holds. We also gave a necessary and sufficient condition for a graph to satisfy the equality hereditarily.

It follows by definition that an independent set is maximal if and only if it is dominating. We improved both the lower and the upper bound on the maximum number of k-dominating independent sets that a graph on n vertices can contain.

We also considered versions of hypergraph domination (k-wise and distance-l domination) and obtained results on how big the vertex set of an r-uniform hypergraph should be if its domination number is at least g. For k-wise domination our results are quite sharp, even the order of magnitude of the error term is determined. The construction for upper bound uses projective geometries.

We considered the domination game on r-uniform hypergraphs and showed a construction of a hypergraph H that asymptotically maximizes (as r tends to infinity) the ratio of the game domination number and the size of the vertex set of H. This ratio is at most 5/9 for 3-uniform hypergraphs.

We also studied the related games on a more general level. Namely we introduced the transversal game on hypergraphs. It can model both the domination game and total domination game played on graphs, via closed- or open neighborhood hypergraphs of graphs. We gave a sharp general upper bound on the parameter called game transversal number, and characterized the hypergraphs attaining maximum. As a corollary, we derived that the 3/4-conjecture on the total domination game is true for every graph of minimum degree at least 2. We also proved an upper bound for the game transversal number of graphs. We further proved that the game transversal number of a 3-uniform hypergraph is at most 5(n+m)/16; for 4-uniform hypergraphs the upper bound is 71(n+m)/252.

VARIANTS OF BIN PACKING PROBLEMS: GAME AND ONLINE

In a series of five papers we studied a class of bin packing games where, beside its size, each item has a positive weight. The payoff associated with an item is the ratio between its weight and the total weight of items packed with it. One special case is when the weight of each item is unit. Every item is selfish: if it can move to another bin where its cost will be smaller, it moves. We studied standard Nash / strong / strictly Pareto optimal / weakly Pareto optimal equilibria (where no more move is possible). We showed that any game of this class admits all these types of equilibria. We studied the (asymptotic) prices of anarchy and stability (PoA and PoS) with respect to these four types of equilibria, for the two cases of general weights and of unit weights. We showed that for general weights all the four PoA values are equal to 1.7, but this is not true for unit weights. Namely, all of them are strictly below 1.7, the strong PoA is approximately 1.691 (well-known number in bin packing) while the strictly Pareto optimal PoA is much lower. We showed that all but one PoS values are equal to 1, the exception is for strong equilibria, which is equal to 1.7 for general weights, and is approximately 1.611824 for unit weights. We also found tight bounds on the worst-case number of steps in processes of convergence to pure Nash equilibria. We also introduced and investigated a new measure for equlibrium (IPoA). We studied how to chose the weights of items to ensure a low PoA.

Further, we introduced a model which generalizes all currently known bin packing games: the payoff of items is determined according to an "interest matrix". Here, additionally to the sizes of the items, an interest matrix with rational entries is also given, its elements stand for how much the items like each other. Each item wants to stay in a bin where it fits and it likes its bin-mates as much as possible. We found that if the matrix is symmetric, a Nash Equilibrium (NE) always exists. However the Price of Anarchy (PoA) may be very large, therefore we considered several special cases and gave bounds for PoA. We presented some results for the asymmetric case, too. Moreover we introduced a new metric, called the Price of Harmony, which we think is more accurate to describe the quality of an NE in the new model.

In the classic online scenario, the items are presented one by one, to be packed into the bins, such that every item is assigned to a bin before the next item is presented. Beside online partitioning of items into classes based on sizes as in previous work, we also applied a new method. This lead us to the design and analysis of an algorithm AH (Advanced Harmonic) whose asymptotic competitive ratio does not exceed 1.5783.

We considered several previously studied online variants of bin packing and proved new and improved lower bounds on the asymptotic competitive ratios for them. We used a method of fully adaptive constructions. In particular, we improved the lower bound for the asymptotic competitive ratio of online square packing significantly, raising it from roughly 1.68 to above 1.75.

In cardinality-constrained bin packing the additional restriction is that each bin may receive at most k items. We resolved the online problem in the sense that we proved a lower bound of 2 on the overall asymptotic competitive ratio. Our result closes this long-standing open problem, since an algorithm of an absolute competitive ratio 2 is known. Additionally, we significantly improved the known lower bounds on the asymptotic competitive ratio for every specific value of k. The novelty of our constructions is based on full adaptivity that creates large gaps between item sizes.

We studied the First Fit bin packing algorithm for its cardinality constrained version, and presented a complete analysis of its asymptotic approximation ratio for all values of k. Many years after FF for BPCC was introduced, its tight asymptotic approximation ratio is finally found.

A very general kind of packing problems is Graph-Bin-Packing, for which we considered game versions. A mapping f is required from the vertex set of an input graph G into a fixed host graph H; among other conditions, for each pair u,v of adjacent vertices of G, the distance of f(u) and f(v) in H must be between two prescribed bounds. We proposed two online versions. The vertices can arrive in any order where each new vertex is adjacent to some previous ones. One version is a Maker-Breaker game whose rules keep the packing conditions. A subclass of Maker-win input graphs G is such that a packing of G is obtained whenever the mapping f is generated by selecting any feasible vertex of H for the next vertex of G in each step. The other model is connected-online packing where we gave an online algorithm which can always find a feasible packing. In both models we obtained some sufficient and some necessary conditions for packability. In the connected-online version we also gave bounds on the size of used part of the host graph.

SCHEDULING

We introduced the new problem “Multiprofessor Scheduling”. It generalizes a number of previous models. It includes two types of conditions, one specifying which of n lectures can be delivered by which of m professors, the other requiring that some professors must be present at some lectures held by others. The optimization problem asks for the shortest possible time within which all lectures can be delivered. For unit-time lectures we proved that the optimum value and an optimal schedule can be determined in polynomial time as a function of n if m is fixed, or in time linear in m if n is fixed; but is NP-complete if both m and n are unbounded. For bounded integer durations of lectures we gave a polynomial-time approximation algorithm. We also introduced and studied a variant that we call interruptive scheduling. It is more restricted than preemptive, and less restricted than non-preemptive.

We generalized Multiprofessor Scheduling to a model which describes that a team works together to process any job. We studied a more restricted version where each job can only be processed on a certain subset of the machines, each job requires a set of renewable resources, and any resource can be used by only one job at a time. The objective is to minimize the makespan. We gave c-approximation algorithms (c constant) for jobs each requiring a bounded number of resources; and proved that the problem is APX-hard, even on three machines with unit-time jobs. For some special cases optimal algorithms with polynomial running time were given.

We considered a semi-online version of the problem of scheduling a sequence of jobs of different lengths on two uniform machines, i.e. with given speeds 1 and s. The optimal offline makespan is known in advance, and the objective is to minimize the makespan. The goal is to determine the optimal competitive ratio, that is, the worst-case ratio of the solution given by an algorithm in comparison to the optimal offline solution. We determined the optimal competitive ratio for speed between 1.7103 and 1.7321, one of the intervals that were still unsolved, by designing and analyzing a compound algorithm.

GRAPH LABELINGS AND GAMES

We initiated the study of many new kinds of graph labeling games. In fact this seems to be the first attempt to propose a systematic study of games in the area of graph labeling.

A graph with the labels 1,2,...,m on its m edges is antimagic if all sums of labels at the vertices are distinct. An old conjecture of Hartsfield and Ringel is that all connected graphs but K_2 are antimagic. This was verified for many graph classes by numerous researchers, but still is open e.g. for trees. We proved that regular (not necessarily connected) graphs are antimagic.

MAJORITY SEARCH AND RELATED PROBLEMS

Majority problems were considered in combinatorial search theory. Given n indexed balls in an urn each having a color unknown to us, the aim is to use the minimum number of queries to find a majority element (a ball in the color class which contains more than half of the balls) or to prove that no such element exists. In the original setting studied first in the 1980s, queries are pairs of balls and an answer tells us whether these balls have the same color. We considered the generalization when a query Q is a subset of the balls, and the answer is a majority element of Q if it exists (or NO if there is no such ball). We studied the 2-color case with query size 3 (some of our results extend to larger query sizes). We gave a linear-time adaptive algorithm and determined the order of magnitude of the minimum number of queries in the non-adaptive case. Other degree-related parameters of query sets were also studied.

For non-fixed number of colors, n indexed balls are given, each colored in some way unknown to us. We should find a ball of the majority color or show that there is no majority color, by asking subsets of [n], that we call queries. A ball i is called majority ball if there are more than n/2 balls in the input set that have the same color as i. Concerning the minimum number of queries, we presented improvements of results of De Marco and Kranakis on different majority models.

The Plurality problem introduced by Aigner has many variants. We dealt with the version where n balls are given, each of them colored by one of three colors. A plurality ball is one whose color class is strictly larger than any other color class. Questioner wants to find a plurality ball as soon as possible by asking triplets, while Adversary partitions the triplets into color classes as an answer for the queries and wants to postpone the possibility of determining a plurality ball. On number $A_p(n)$ of queries needed to ask if both play optimally we provided an almost precise result by proving that for $n \geq 4$ even we have $3n/4-2 \leq A_p(n) \leq 3n/4-1/2$, and for $n \geq 3$ odd we have $3n/4-O(\log n) \leq A_p(n) \leq 3n/4-1/2$.

Another search problem is: some excellent elements of [n] are given, and we should find at least one, asking questions of the type „Is there an excellent element in subset A of [n]?". We verified a conjecture of G.O.H. Katona by proving that for fixed r the r-round version needs $rn^{1/r}+O(1)$ queries, and this bound is sharp.

In combinatorial group testing the problem is to find a special element x in [n] by testing subsets of [n]. Tapolcai et al. introduced a new model where each element knows the answer for those queries that contain it and each element should be able to identify the special one. We proved that if a set system F_n solves the non-adaptive version of this problem on [n] and has minimum cardinality, then $\lim_{n\to\infty} |F_n|/\log_2 n=\log_{3/2} 2$. This improves results by Tapolcai et al. We studied similar further problems.

SOME EXTREMAL THEOREMS

We studied a very general combinatorial problem, namely the minimum number of r-element subsets such that every k-partition of an n-element set is `crossed' by at least one of them. We proved asymptotically tight estimates and pointed out connections with Turán-type extremal problems on graphs and hypergraphs, and with balanced incomplete block designs.

The forbidden subposet problems ask for the maximum size of a subfamily F of the power set of an n-element set provided F does not contain a certain fixed inclusion pattern. We reproved some asymptotically optimal bounds of known results but under the weaker assumption that only those occurrences of the inclusion pattern (described by a poset P) are forbidden in F in which elements of P with same rank are represented with sets of same size.

We obtained an Erdős-Stone-Simonovits type theorem on the maximum number of k-chains in a P-free family. It states that the degeneracy depends on the height of the forbidden subposet P.

Forbidden inclusion patterns can be described by posets. We proved several Hilton-Milner type results on the "second largest" families (vertex subsets of the Kneser graph) when the forbidden intersection pattern is described by complete graphs, complete multipartite graphs, cycles or paths.

Edge or vertex Turán problems have been investigated with the hypercube as host graph. It turns out that if we consider the hypercube with the edges oriented towards the endpoint corresponding to the larger set, then the corresponding Turán problems have a flavor similar to forbidden subposet problems. We answered some instances of the Turán problem in the oriented variant and also obtained a result in the original non-oriented case.

We dealt with the (vertex) ordered variant of the classical Turán problem. The result of Bondy and Simonovits states that if a graph on n vertices contains no cycle of length 2k then it has at most $O(n^{1+1/k})$ edges. However, matching lower bounds are only known for k=2,3,5. We showed that the maximum number of edges in a vertex ordered graph avoiding so called bordered cycles of length at most 2k is $\Theta(n^{1+1/k})$. Strengthening the result of Bondy and Simonovits in the case of 6-cycles, we also showed that it is enough to forbid these bordered orderings of the 6-cycle to guarantee an upper bound of $O(n^{4/3})$ on the number of edges.

We asymptotically determined the maximum number of hyperedges possible in an r-uniform, connected n-vertex hypergraph without a Berge path of length k, as n and k tend to infinity. We showed that, unlike in the graph case, the multiplicative constant is smaller with the assumption of connectivity.

Given two graphs H and F, the maximum possible number of copies of H in an F-free graph on n vertices is denoted by ex(n,H,F). We investigated the function ex(n,H,kF), where kF denotes k vertex disjoint copies of a fixed graph F. Our results include cases where F is a complete graph, cycle or a complete bipartite graph.

We proved various results concerning the number of even cycles in a graph, under the assumption that some set of cycles is forbidden.

We investigated the discrete Fuglede's conjecture and Pompeiu problem on finite abelian groups and developed a strong connection between the two problems. We gave a geometric condition under which a multiset of a finite abelian group has the discrete Pompeiu property. Using this description and the revealed connection we proved that Fuglede's conjecture holds for $Z_{p^n q^2}$, where p and q are different primes. In particular, we showed that every spectral subset of $Z_{p^n q^2}$ tiles the group.

We initiated the general study of the Ramsey problem for Berge hypergraphs, mainly for uniform hypergraphs. We focused on the relation between the number of colors and the uniformity, and determined the Ramsey number for various ranges for complete r-uniform Berge hypergraphs. We also dealt with the Ramsey number of trees.

SOME FURTHER RESULTS

Tilted Sperner families were introduced by Gil Kalai, motivated by a new proof of the density Hales-Jewett theorem. A related notion is tilted Sperner families with patterns, a generalization of Sperner families on the ordered set (1,2,...,n). Applying the well-known permutation method we sharpened an upper bound on the size of such families.

Planarly connected crossing (PCC in short) graphs can be considered as generalizations of planar graphs. We proved that if a graph G can be drawn in the plane such that two crossing edges never share a vertex but there is a crossing-free edge that connects their endpoints, then G has linearly many edges. So, PCC graphs of this kind are not much bigger than planar graphs.

Coloring geometric hypergraphs is a very active topic with lots of applications even outside mathematics. We proved that there is a constant k such that any finite set S of points in the plane can be 2-colored such that every axis-parallel square containing at least k points from S contains points of both colors. Our proof is constructive and provides a polynomial-time algorithm for obtaining such a 2-coloring. By affine transformations this result applies also for homothets of a fixed parallelogram.

Results for factor of IID processes on regular trees have lots of applications in the theory of regular graphs (bounds on the size of maximal independent sets, etc.). We proved that if such a process is restricted to two distant connected subgraphs of the tree, then the two parts are basically uncorrelated. This result can be considered as a generalization of a result of Bachkhausz, Virág and Szegedy, and also as a quantitative version of the fact that factor of IID processes have trivial 1-ended tails, which gives a quantitative version of a result of R. Lyons.

Some types of percolation can be thought of as a one-person game: vertices of a graph get infected in a discrete time process according to the number of already infected neighbor vertices. Standard questions are: the minimum number of vertices needed to be infected so that the process percolates (all vertices get infected by the end); critical probability of selecting infected vertices randomly so that the process percolates; maximum time in which percolation may occur; other parameters. Recently, Balister et al. introduced a new version called line percolation. We generalized it as follows: given a hypergraph H and an integer r, an uninfected vertex v gets infected if there exsts an edge e containing v such that at least r vertices of e are already infected. We obtained many sharp results for the parameters above in the case when H is a finite projective plane.

We characterized the graphs whose triangle graph is a cycle. Moreover, for each n>2 we gave forbidden subgraph characterization for graphs whose triangle graph is C_n-free. As a consequence, we characterized graphs whose triangle graph is a tree or perfect or chordal graph. These are related to the study of minimum edge covers of triangles of a graph.