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Given a finite n-element set X, a family of subsets $\mathcal{F} \subset 2^X$ is said to *separate* X if any two elements of X are separated by at least one member of $\mathcal{F}$. It is shown that if $|\mathcal{F}| > 2^{n-1}$, then one can select $\lceil \log n \rceil + 1$ members of $\mathcal{F}$ that separate X. If $|\mathcal{F}| \geq \alpha 2^n$ for some $0 < \alpha < 1/2$, then $\log n + O(\log \frac{1}{\alpha} \log\log \frac{1}{\alpha})$ members of $\mathcal{F}$ are always sufficient to separate all pairs of elements of X that are separated by some member of $\mathcal{F}$. This result is generalized to simultaneous separation in several sets. Analogous questions on separation by families of bounded Vapnik-Chervonenkis dimension and separation of point sets in $\mathbb{R}^d$ by convex sets are also considered [7].

The *disjointness graph* $G = G(\mathcal{S})$ of a set of segments $\mathcal{S}$ in R^d, $d \geq 2$, is a graph whose vertex set is $\mathcal{S}$ and two vertices are connected by an edge if and only if the corresponding segments are disjoint. We prove that the chromatic number of G satisfies $\chi(G) \leq (\omega(G))^4 + (\omega(G))^3$, where $\omega(G)$ denotes the clique number of G. It follows, that $\mathcal{S}$ has $\Omega(n^{1/5})$ pairwise intersecting or pairwise disjoint elements. Stronger bounds are established for lines in space, instead of segments [15].

We also show that computing $\omega(G)$ and $\chi(G)$ for disjointness graphs of lines in space are NP-hard tasks. However, we can design efficient algorithms to compute proper colorings of G in which the number of colors satisfies the above upper bounds. One cannot expect similar results for sets of continuous arcs, instead of segments, even in the plane. We construct families of arcs whose disjointness graphs are triangle-free ($\omega(G) = 2$), but whose chromatic numbers are arbitrarily large.

A combinatorial geometric question is considered in [32]. Let $h(n)$ denote the maximum number of triangles with angles between 59° and 61° in any n-element planar set. We also prove $h(n) = n^3/24 + O(n \log n)$ as $n \to \infty$. However, there are triangles T and n-point sets P showing that the number of ε-similar copies of T in P can exceed $n^3/15$ for any $\varepsilon > 0$.

Let G be a fixed graph. Two paths of length $n-1$ on n vertices (Hamiltonian paths) are G-different if there is a subgraph isomorphic to G in their union. In this paper we prove that the maximal number of pairwise triangle-different Hamiltonian paths is equal to the number of balanced bipartitions of the ground set, answering a question of Körner, Messuti and Simonyi [23].

Assume that no hyperplane intersects the curve C in $\mathbb{R}^d$ more than $d+1$ times. Then C can be split into $M = M(d)$ convex curves (where $M(d)$ depends only on d). This is the main result in [4]. It implies that a certain geometric Ramsey number is a d-fold tower of n, a breakthrough result in Ramsey theory.

We show in [20] that, as a consequence of a new result of Pór on universal Tverberg partitions, any large-enough set P of points in $\Re^d$ has a $(d+2)$-sized subset whose Radon point has half-space depth at least $c_d \cdot |P|$, where $c_d \in (0,1)$ depends only on d. We then give an application of this result to computing weak ε-nets by random sampling. In particular, we show that given any set P of points in $\Re^d$ and a parameter $\varepsilon > 0$, there exists a set of $O\left(\frac{1}{\varepsilon^{\lfloor \frac{d}{2} \rfloor + 1}}\right)$ $\lfloor \frac{d}{2} \rfloor + 1$-dimensional simplices (ignoring polylogarithmic factors) spanned by points of P such that they form a transversal for all convex objects containing at least $\varepsilon \cdot |P|$ points of P.

[14] is a thorough survey presenting an overview of the advances around Tverberg's theorem, focusing mainly on the last two decades. We discuss the topological, linear-algebraic, and combinatorial aspects of Tverberg's theorem and its applications. The survey contains several open problems and conjectures.

The paper [28] gives a no-dimensional version of Carathédory's theorem: given an n-element set $P \subset \Re^d$, a point $a \in \text{conv}P$, and an integer $r \leq d$, $r \leq n$, there is a subset $Q \subset P$ of r elements such that the distance between a and $\text{conv}Q$ is less than $\text{diam}P/\sqrt{2r}$. The similar no-dimension Helly theorem says that, given $k \leq d$ and a finite family $\mathcal{F}$ of convex bodies, all contained in the Euclidean unit ball of $\Re^d$, there is a point $q \in \Re^d$ which is closer than $1/\sqrt{k}$ to every set in $\mathcal{F}$. This result has several colourful and fractional consequences. Similar versions of Tverberg's theorem and some of their extensions are also established.

A graph is 1-planar if it can be drawn in the plane such that each edge is crossed at most once. A graph, together with a 1-planar drawing is called 1-plane. A graph is maximal 1-planar (1-plane), if we can not add any missing edge so that the resulting graph is still 1-planar (1-plane). Brandenburg et al. showed that there are maximal 1-planar graphs with only $\frac{45}{17}n + O(1) \approx 2.647n$ edges and maximal 1-plane graphs with only $\frac{7}{3}n + O(1) \approx 2.33n$ edges. On the other hand, they showed that 19] a maximal 1-planar graph has at least $\frac{28}{13}n - O(1) \approx 2.15n - O(1)$ edges, and a maximal 1-plane graph has at least $2.1n - O(1)$ edges. We improve both lower bounds to $\frac{20n}{9} \approx 2.22n$ in [25].

A new direction concerning Tverberg's theorem is initiated in [21]: Given a set $A \subset \mathbb{R}^d$ in general position with $|A| = (r-1)(d+1)+1$ and $k \in \{0, 1, \dots, r-1\}$, there is a partition of A into r sets $A_1, \dots, A_r$ (where $|A_j| \le d+1$ for each j) with the following property. There is a unique $z \in \bigcap_{j=1}^{r} \operatorname{aff} A_j$ and it can be written as an affine combination of the element in A_j: $z = \sum_{x \in A_j} \alpha(x)x$ for every j and exactly k of the coefficients $\alpha(x)$ are negative. The case $k = 0$ is Tverberg's classical theorem.

An integral zonotope in $\mathbb{R}^d$ is the Minkowski sum of finitely many segments $[0, z_i]$ $i \in [n]$ where $z_i \in \mathbb{R}^d$ is an integer vector. Its endpoint is $\sum_1^n z_i$. Given a pointed convex cone C with nonempty interior and an integer vector $z \in C$ let $\mathcal{F}(C, z)$ denote the set of all integral zonotopes whose endpoint is z. The set $\mathcal{F}(C, z)$ is clearly finite; in [19] we give an asymptotic formula for the size of $\mathcal{F}(C, \lambda z)$ when z is fixed and $\lambda \to \infty$. More importantly we show that, again when z is fixed and λ islarge, the zonotopes in $\mathcal{F}(C, \lambda z)$ have a **limit shape**, that is, the overwhelming majority of the zonotopes in $\mathcal{F}(C, \lambda z)$ are very close to λ times a fix convex zonoid, whic is called the limit shape. The proofs combine probablity theory, convex geometry, and the geometry of numbers.

Let $U_1, \dots, U_{d+1}$ be n-element sets in $\mathbb{R}^d$ and let $\langle u_1, \dots, u_{d+1} \rangle$ denote the convex hull of points $u_i \in U_i$ (for all i) which is a simplex. Pach's selection theorem is about such simplices. It says that there are sets $Z_1 \subset U_1, \dots, Z_{d+1} \subset U_{d+1}$ and a point $u \in \mathbb{R}^d$ such that each $|Z_i| \ge c_1(d)n$ and $u \in \langle z_1, \dots, z_{d+1} \rangle$ for every choice of $z_1 \in Z_1, \dots, z_{d+1} \in Z_{d+1}$. In [22] we show that this theorem does not admit a topological extension with linear size sets Z_i. Further we prove a topological extension where each $|Z_i|$ is of order $(\log n)^{1/d}$.

We describe in [6] how a powerful new „constraint method" yields many different extensions of the topological version of Tverberg's 1966 Theorem in the prime power case. The same method also was instrumental in the recent spectacular construction of counterexamples for the general case.

The intrinsic volumes of Gaussian polytopes are considered in [11]. A lower variance bound for these quantities is proved, showing that, under suitable normalization, the variances converge to strictly positive limits. The implications of this missing piece of the jigsaw in the theory of Gaussian polytopes are discussed.

Gershgorin's famous circle theorem states that all eigenvalues of a square matrix lie in disks (called Gershgorin disks) around the diagonal elements. We show in [13] that if the matrix entries are non-negative and an eigenvalue has geometric multiplicity at least two, then this eigenvalue lies in a smaller disk. The proof uses geometric rearrangement inequalities on sums of higher dimensional real vectors which is another new result.

Let $b(M)$ denote the maximal number of disjoint bases in a matroid M. It is shown that if M is a matroid of rank $d+1$, then for any continuous map f from the matroidal complex M into $\mathbb{R}^d$ there exist $t \ge \sqrt{b(M)}/4$ disjoint independent sets $\sigma_1, \dots, \sigma_t \in M$ such that $\bigcap_{i=1}^t f(\sigma_i) \ne \emptyset$. This is an extension of Tverberg's famous thoerem to matroids, and is the main content of [12].

In the article [8] we define an **algebraic vertex** of a generalized polyhedron and show that the set of algebraic vertices is the smallest set of points needed to define the polyhedron. We prove that the indicator function of a generalized polytope P is a linear combination of indicator functions of simplices whose vertices are algebraic vertices of P. We also show that the indicator function of any generalized polyhedron is a linear combination, with integer coefficients, of indicator functions of cones with apices at algebraic vertices and line-cones. The concept of an algebraic vertex is closely related to the Fourier–Laplace transform. We show that a point $\mathbf{v}$ is an algebraic vertex of a generalized polyhedron P if and only if the tangent cone of P, at $\mathbf{v}$, has non-zero Fourier–Laplace transform.

Let P be a star-shaped polygon in the plane, with rational vertices, containing the origin. The number of primitive lattice points in the dilate tP is asymptotically $\frac{6}{\pi^2}\mathrm{Area}(tP)$ as $t \to \infty$. The results in the paper [18] show that the error term is both $\Omega_{\pm}\big(t\sqrt{\log\log t}\big)$ and $O(t(\log t)^{2/3}(\log\log t)^{4/3})$. Both bounds extend (to the above class of polygons) known results for the isosceles right triangle, which appear in the literature as bounds for the error term in the summatory function for Euler's $\phi(n)$.

A random spherical polytope P_n in a spherically convex set $K \subset S^d$ as considered here is the spherical convex hull of n independent, uniformly distributed random points in K. The behaviour of P_n for a spherically convex set K contained in an open halfsphere is quite similar to that of a similarly generated random convex polytope in a Euclidean space, but the case when K is a halfsphere is different. This is what we investigate in [9], establishing the asymptotic behaviour, as n tends to infinity, of the expectation of several characteristics of P_n, such as facet and vertex number, volume and surface area. For the Hausdorff distance from the halfsphere, we obtain also some almost sure asymptotic estimates.

It is a recent result that given a finitely many points on $\mathbb{R}^2$, it is possible to arrange them on a polygonal path so that every angle on the polygonal path is at least $\pi/9$. In [2] we extend this result to finite sets contained in the 2-dimensional sphere.

Let $\|.\|$ be a norm in $\mathbb{R}^d$ whose unit ball is B. Assume that $V \subset B$ is a finite set of cardinality n, with $\sum_{v\in V} v = 0$. We show in [1] that for every integer k with $0 \le k \le n$, there exists a subset U of V consisting of k elements such that $\|\sum_{v\in U} v\| \le \lceil d/2 \rceil$. We also prove that this bound is sharp in general. We improve the estimate to $O(\sqrt{d})$ for the Euclidean and the max norms. An application on vector sums in the plane is also given.

Given n continuous open curves in the plane, we say that a pair is *touching* if they have finitely many interior points in common and at these points the first curve does not get from one side of the second curve to its other side. Otherwise, if the two curves intersect, they are said to form a *crossing* pair. Let t and c denote the number of touching pairs and crossing pairs, respectively. We prove that $c \ge \frac{1}{10^5}\frac{t^2}{n^2}$, provided that $t \ge 10n$. Apart from the values of the constants, this result is best possible [26].

Let G be a drawing of a graph with n vertices and $e > 4n$ edges, in which no two adjacent edges cross and any pair of independent edges cross at most once. According to the celebrated Crossing Lemma of Ajtai, Chvátal, Newborn, Szemerédi and Leighton, the number of crossings in G is at least $c\frac{e^3}{n^2}$, for a suitable constant $c > 0$. In a seminal paper, Székely generalized this result to multigraphs, establishing the lower bound $c\frac{e^3}{mn^2}$, where m denotes the maximum multiplicity of an edge in G. In [27] we get rid of the dependence on m by showing that, as in the original Crossing Lemma, the number of crossings is at least $c'\frac{e^3}{n^2}$ for some $c' > 0$, provided that the „lens" enclosed by every pair of parallel edges in G contains at least one vertex. This settles a conjecture of Bekos, Kaufmann, and Raftopoulou.

The situation turns out to be quite different if nonparallel edges are allowed to cross any number of times. It is proved in [31] that in this case the number of crossings in G is at least $c''e^{2.5}/n^{1.5}$. The order of magnitude of this bound cannot be improved.

The *crossing number* $\mathrm{cr}(G)$ of a graph $G = (V, E)$ is the smallest number of edge crossings over all drawings of G in the plane. For any $k \ge 1$, the *k-planar crossing number* of G, $\mathrm{cr}_k(G)$, is defined as the minimum of $\mathrm{cr}(G_0) + \mathrm{cr}(G_1) + \ldots + \mathrm{cr}(G_{k-1})$ over all graphs $G_0, G_1, \ldots, G_{k-1}$ with $\cup_{i=0}^{k-1} G_i = G$. It is shown in [29] that for every $k \ge 1$, we have $\mathrm{cr}_k(G) \le \big(\frac{2}{k^2} - \frac{1}{k^3}\big)\,\mathrm{cr}(G)$. This bound does not remain true if we replace the constant $\frac{2}{k^2} - \frac{1}{k^3}$ by any number smaller than $\frac{1}{k^2}$. Some of the results extend to the rectilinear variants of the k-planar crossing number.

A convex polygon Q is *circumscribed* about a convex polygon P if every vertex of P lies on at least one side of Q. We present an algorithm for finding a maximum area convex polygon circumscribed about any given convex polygon in $O(n^3)$ time [30]. As an application, we disprove a conjecture of Farris. Moreover, for the special case of regular n-gons we find an explicit solution. Our results can be used to bound the integral of a convex function. This is has an important application related to the Gini index in statistics.

A planar point set of n points is called γ-*dense* if the ratio of the largest and smallest distances among the points is at most $\gamma\sqrt{n}$. In [24] we construct a dense point set of n points in the plane with $ne^{\Omega\left(\sqrt{\log n}\right)}$ halving lines. This improves the bound $O(n \log n)$ of Edelsbrunner, Valtr and Welzl from 1997.

Our construction can be generalized to higher dimensions, for any d we construct a dense point set of n points in R^d with $n^{d-1}e^{\Omega\left(\sqrt{\log n}\right)}$ halving hyperplanes. Our lower bounds are asymptotically the same as the best known lower bounds for general point sets.

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