Detailed report on the results of the OTKA/NKFIH grant PD 108406

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Coloring Geometric Hypergraphs

In accordance with my research proposal, the main body of my research is concentrated on geometric hypergraph colorings and cover-decomposability problems. I managed to make substantial progress in the area together with my coauthors.

As a starting point, earlier, with Pálvölgyi we proved that translates of octants in the three dimensional space are cover-decomposable into two coverings. We managed to generalize this to decomposition into k coverings [4], which work was already done partially during the grant. In a subsequent paper we also managed to improve the bounds on the constant for the cover-decomposability of translates of octants for two coverings [6]. This immediately improves the bounds for decomposition into k coverings. We have shown earlier that results about translates of octants imply the same about homothets of triangles. Thus, these new improved bounds about octants also apply to the decompositions of the homothets of a triangle and also to the dual problem for triangles.

As our proofs about octants are based on the fact that this problem is equivalent to a certain quasi-online coloring of quadrants in the plane, it is interesting to understand better what level of 'onlineness' is enough for such a statement to be true. With Lemons and Pálvölgyi we proved [7] that the usual online version of this statement is false and we gave good bounds on the number of colors needed in this case. We also gave similar bounds for the case of online coloring intervals, which is important, as intervals are always the classic starting points of algorithmic coloring problems. In [7] we also revisited an older result of mine about coloring bottomless rectangles, or equivalently, about quasi-online coloring intervals.

For simplicity, we omitted the definition of cover-decomposability. However, we do define the dual problem, as it is slightly simpler and also plays an important role in the results we are about to detail. The general problem is the following. Given a convex polygon P, we need to decide if there is a constant m_k such that any finite set of points can be colored with k colors such that any homothetic copy of P with at least m_k points contains all k colors. We note that in general this is not equivalent to the corresponding cover-decomposability problem. Yet, if instead of homothets we consider translates of P (as was the case in most of the research done earlier in this area) then this dual problem is equivalent to the cover-decomposability problem.

Building on our result in [4], for the dual case for the homothets of triangles, Cardinal et al. improved the dependence on k of the upper bound from double exponential to a polynomial, proving $m_k \leq ck^{7.17}$. We improved further this method, introducing the notion of self-coverability of polygons, and proved a more general result [5], which in turn improves also this bound to $m_k \leq ck^{4.53}$. Following similar ideas, a (slightly worse) polynomial bound was found for translates of octants as well (which works in the primal and dual case as well) by Cardinal et al.

Our result in [5] states that if for some polygon m_2 exists then m_k also exists and is a polynomial of k. Note that to apply the theorem we need a polygon for which m_2 exists. Remarkably, for the primal problem, i.e., for cover-decomposing, the corresponding question turns out to be false for every polygon except the triangle, as proved by I. Kovács. This makes even more intriguing the question if for squares (as the simplest polygon that is not a triangle) m_2 does exist. As the culmination of my research, with Ackerman and Vizer [13] we proved that m_2 does indeed exist. That is, we proved that points with respect to squares are 2-(and also k)-colorable, using many tools, e.g., 4-colorability of generalized Delaunay triangulations and most importantly our earlier introduced notion of self-coverability turned out to play an essential role in this proof. It is an important aspect of this result that this is perhaps the first case when the primal and dual problem has a completely different answer.

It was an important progress in the area when Pálvölgyi proved that the translates of a disk in the plane are not cover-decomposable. In the same paper he proves that the translates of an inifinite convex shape are cover-decomposable to two coverings and asks what is the case for decomposing into many coverings. With him we managed to prove optimal bounds for this problem [10]. In fact we proved the corresponding much more general statement about decomposing pseudo-halfplanes. The proof method is based on the (much simpler) earlier proof of Smorodinsky and Yuditsky for the case of decomposing halfplanes. For pseudo-halfplanes our bounds are optimal for the dual case of coloring points with respect to half-planes but slightly worse than optimal for decomposing pseudo-halfplanes. Besides the new results, we introduced a purely abstract approach to consider such geometric problems (defining ABA-free hypergraphs among others) which might be a useful tool to tackle further related problems.

Conflict-free coloring is a notion with real life applications and which is directly connected to the geometric hypergraph coloring problems considered during my grant. Final stages of a work on a result about conflict-free coloring abstract hypergraphs [1], which is a combinatorial counterpart of conflict-free coloring geometric hypergraphs was done during the grant.

These results appeared or will appear in leading journals of the area (CGTA, DCG, etc.), while most of them were also presented in prestigious combinatorial (Sum(m)it:240 2014, Intuitive Geometry 2015) and computer science conferences with peer-reviewed proceedings (SoCG 2016, WG 2015).

The main area of my research, coloring geometric hypergraphs, is in the intersection of the areas of geometry, combinatorics and algorithmic problems. The next two sections cover the results in the various intersections of these areas that are related less closely to the main topic.

Other geometric results

An important algorithmic problem in geometry is pattern matching. With Ben-Avraham, Henze, Jaume, Raz, Sharir and Tubis we considered [17] the problem when in a big point set (of size n) we need to find a subset of points that resembles most a given pattern, which is a small set of points (of size m), in our case the pattern can be only translated. Although the problem is not known to be polynomial, we established several structural properties of the underlying subdivision of the plane and derived improved bounds on its complexity. Specifically, we showed that this complexity is $O(n^2 m^{3.5}(e \ln m + e)^m)$, so it is only quadratic in the size of the big point set. This is much better than the trivial upper bound of $O(n^m)$. An important tool was a combinatorial matching result which we proved with Asinowski and Miltzow [9]. This result is interesting on its own as well, as it considers a problem that can occur in real life situtations in economics.

In graph drawing it is a usual setting that given a family of graphs (defined by some geometric restriction) we are interested in the maximal number of edges such a graph can have. With Ackerman and Vizer we proved in [14] that a certain such class, the family of planarly-connected crossing graphs, has linear many edges. This can be considered as a step toward proving that for some related, widely investigated classes (quasi-planar graphs in particular) the answer might be the same.

In [2] I proved simple results about covering grid-like planar point sets with polygonal lines and trees, which spawned further results by other researchers for the corresponding problem in higher dimensions.

Besides journal publications, these results were also presented in conferences (ESA 2014, FUN 2014, EuroCG 2014, Graph Drawing 2016).

Other combinatorial and algorithmic results

Most importantly with Győri we proved [15] a 25 year old conjecture which is a special case of a conjecture of Erdős. The extremal combinatorial result we proved is that if a K_4-free graph has k more edges then the maximal balanced bipartite graph on the same set of vertices, then it contains k edge-disjoint triangles.

With Patkós and Zhu [3] we investigated nonrepetitive colorings of lexicographic product of paths and other graphs, where we managed to prove good bounds for the needed number of colors. Later with Zhu we also considered the even more natural problem of the choosability and online choosability (paintability) of the lexicographic product of graphs, which result is written as a manuscript as of now, submitted to a journal, and so it is not mentioned in the publication list.

Next are three papers about combinatorial search. First, with Gerbner, Pálvölgyi, Rote and Wiener [11] we proved lower and upper bounds about the minimal number of queries to find an endpoint of a path when there is a known graph with unknown disjoint underlying paths and cycles and a query is a vertex of the graph. In particular, showing a strong connection between this problem and graph separators, we provide almost tight bounds in case the graph is a grid graph. Second, with Cicalese, Lidicky, Pálvölgyi and Valla [16] we improved an earlier result of one of the authors by presenting a more efficient approximation algorithm on the tree search problem with non-uniform costs. Third, with Gerbner, Pálvölgyi, Patkós, Wiener and Vizer [8] we showed several results about variants of the problem of finding a majority ball with majority answers.

With Gerbner, Palmer and Pálvölgyi in [12] we considered a problem about weighted directed acyclic graphs. We showed that a process that marks every vertex at the end by marking and unmarking, such that during the process the marked nodes always form a downwards closed set (the directed edges are imagined as going upwards) and the sum of their weights is always nonnegative, can be always simplified to a purely marking process of the same kind.

Two further results are in the preparation stage and thus not included in the publication list.

Besides journal publications, these results were also presented in conferences (Eurocomb 2015, WG 2015, Joint Austrian-Hungarian Mathematical Conference 2015, Hungarian-Israeli combinatorial days at the Technion 2016, etc.).

Final remarks

During my grant I was invited by Rom Pinchasi to Technion, Haifa, where for a month we have been working with him and Eyal Ackerman on various combinatorial geometric problems. In order to continue this research, I used parts of my budget on inviting Eyal Ackerman, and with the additional help of other resources, Eyal could come to Hungary to be one of the senior researchers of the 7th Emléktábla Workshop, which I also attended. Afterwards, he stayed for approx. 3 weeks in July 2015, during which time we did intensive work with him and Máté Vizer. The most important parts of the research related to the papers [13,14] were done during this time.

During the grant period I have regularly attended the Emléktábla Workshop. This workshop series was also partially supported by the grant. In 2014 I was invited to participate in the Oberwolfach Workshop on Discrete Geometry.

In 2015 I've been awarded the Junior Prima Prize of the Hungarian Academy of Sciences.