Final Report

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The main topic of the project was the study of the decomposition of multiple coverings, initiated by J. Pach in 1980. The project ended with a breakthrough, giving a negative answer to the main question posed by Pach. We have also completed a survey of the topic [11].

Basics of Cover Decomposition

Let $\mathcal{P} = \{\ P_i \mid i \in I\ \}$ be a collection of planar sets. We say that $\mathcal{P}$ is an *m-fold covering* if every point in the plane is contained in at least m members of $\mathcal{P}$. The biggest such k is called the thickness of the covering. A 1-fold covering is simply called a *covering*.

Definition. A planar set P is said to be *cover-decomposable* if there exists a (minimal) constant $m = m(P)$ such that every m-fold covering of the plane with translates of P can be decomposed into two coverings.

Pach [9] proposed the problem of determining all cover-decomposable sets in 1980 and made the following conjecture.

Conjecture. (Pach) *All planar convex sets are cover-decomposable.*

We have managed to disprove this conjecture by exhibiting for any m an m-fold covering of the plane with unit disks that cannot be decomposed into two disjoint coverings, i.e., no matter how one colors the unit disks of the covering with two colors, one of the color classes will not cover the whole plane [10]. The construction easily generalizes to other covering shapes and regions to be covered. The main parts of the construction are explained at the end of this report.

Octants

Note that earlier the problem was mainly investigated for polygons.

For a cover-decomposable set P, one can ask for the exact value of $m(P)$. In most of the cases, the best known upper and lower bounds are very far from each other. The only case where the gap is relatively small is for open triangles where we have proved with Keszegh that $4 \leq m(P) \leq 9$ [8] improving our earlier result [4]. This result is a corollary of a more general theorem about coverings of the three dimensional space by the translates of an octant. Here the notions can be similarly defined and an octant is the natural generalization of a quadrant, e.g., the set $(0, \infty) \times (0, \infty) \times (0, \infty)$. We have in fact proved that $5 \leq m(\mathit{octant}) \leq 9$.

Decomposition to multiple parts

Definition. Let P be a planar set and $k \geq 2$ integer. If it exists, let $m_k(P)$ denote the smallest number m with the property that every m-fold covering of the plane with translates of P can be decomposed into k coverings.

The number $m_k(P)$ might be finite for all cover-decomposable P, moreover, maybe even $m_k(P) = O(m_2(P))$ holds. This was established for polygons by Gibson and Varadarajan [2], i.e., they have proved that for any open convex polygon P, $m_k(P) = O(k)$.

For octants, first we could show that $m_k(\mathit{octant})$ is finite [5]. Later this was improved to polynomial by Cardinal et al. [1], who used our newly developed general method that allows one to show that $m_k(P)$ is polynomial given certain conditions [6].

We have managed to prove similar results related to pseudohalfplane arrangements [7], generalizing results of Smorodinsky and Yuditsky [14]. Our approach is entirely abstract and builds on a generalization of *shift-chains*, which I have defined in my PhD thesis [12].

We have proved some related results about so-called *online* colorings in [3].

Construction - indecomposable covering by unit disks

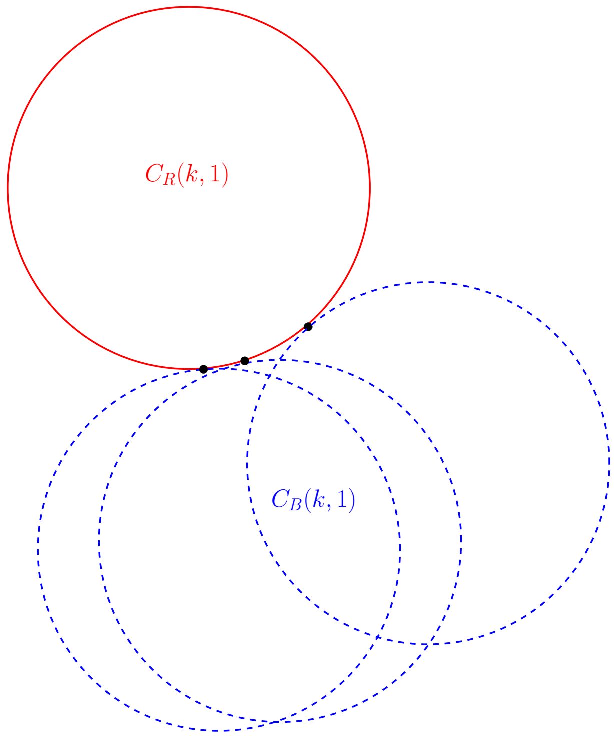
Here we sketch the main idea of the counterexample for the conjecture if instead of the whole plane, only a finite point set is being covered. In the equivalent dual (here we do not go into details about why this equivalence holds) this means that we have to give, for every k, a finite point set, S, such that in any 2-coloring of S there is a unitdisk that contains exactly k points of S and each of these have the same color. The construction is the realization of the same hypergraph that was realized in [13] with a similar induction argument. First we describe the hypergraph $\mathcal{H}(k,l)$ which is constructed inductively.

The set of vertices of $\mathcal{H}(k,l)$, denoted by $V(k,l)$, is defined inductively as $V(k-1,l) \cup^* V(k,l-1) \cup^* \{p\}$ if k and l are both bigger than 1 and as kl points if k or l equals 1. The set of edges of $\mathcal{H}(k,l)$ are given as the disjoint union of a k-uniform set, $E_R(k,l)$, and an l-uniform set, $E_B(k,l)$, which are defined as follows. If $l=1$, then $E_R(k,1) = \{V(k,1)\}$ and $E_B(k,1) = V(k,1)$. Similarly, if $k=1$, then $E_R(1,l) = V(1,l)$ and $E_B(1,l) = \{V(1,l)\}$. If k and l are both bigger than 1, then $E_R(k,l) = \{e \cup \{p\} \mid e \in E_R(k-1,l)\} \cup E_R(k,l-1)$, and $E_B(k,l) = \{e \cup \{p\} \mid e \in E_B(k,l-1)\} \cup E_B(k-1,l)$. A simple inductive argument (see [13]) gives

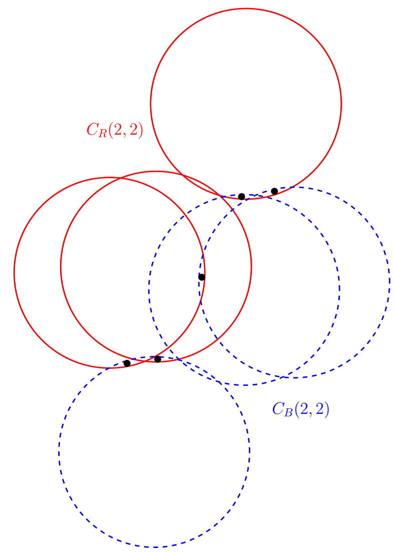
Lemma. *$\mathcal{H}(k,l)$ cannot be 2-colored, if we color its vertices with red and blue, then either there is an edge in $E_R(k,l)$ that contains only k red points, or there is an edge in $E_B(k,l)$ that contains only l blue points.* $\square$

Now our goal is to realize $\mathcal{H}(k,l)$ with unitdisks, i.e., map $V(k,l)$ into different points in the plane, $S(k,l)$, such that for any $e \in E_R(k,l) \cup E_B(k,l)$ there is a unitdisk that contains exactly the points that correspond to the elements of e. The dual of this construction for $k=l$ will give an indecomposable, k-fold covering of a finite point set. The realization is also done by induction.

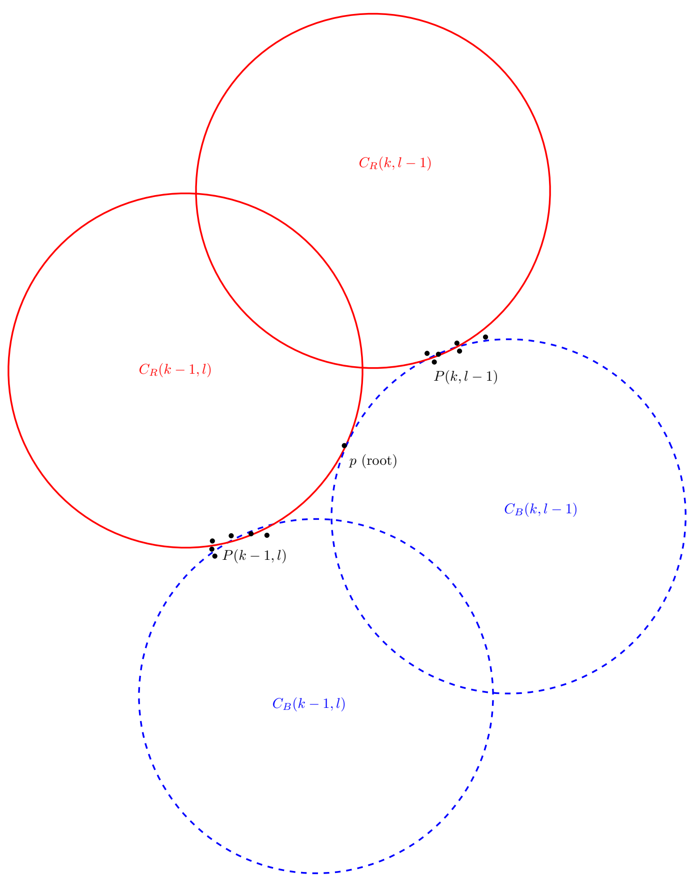
In the realization, for each edge we select (and fix) a realizing unitdisk, and depending on the type of the realized edge, we partition these discs into two collections, $\mathcal{C}_R(k,l)$ and $\mathcal{C}_B(k,l)$. For any k and l, we will have for some small $\varepsilon(k,l) > 0$ that (omitting the parameters k and l whenever it leads to no confusion) $d(C,C') < \varepsilon$ if C and C' both belong to $\mathcal{C}_R$ or both belong to $\mathcal{C}_B$, while $2-\varepsilon < d(C,C') < 2$ if one is in $\mathcal{C}_R$, the other in $\mathcal{C}_B$. (This means that from “far”, the two families



(a) Starting step



(b) $\mathcal{H}(2,2)$ magnified



(c) Induction step

Figure 1: The construction

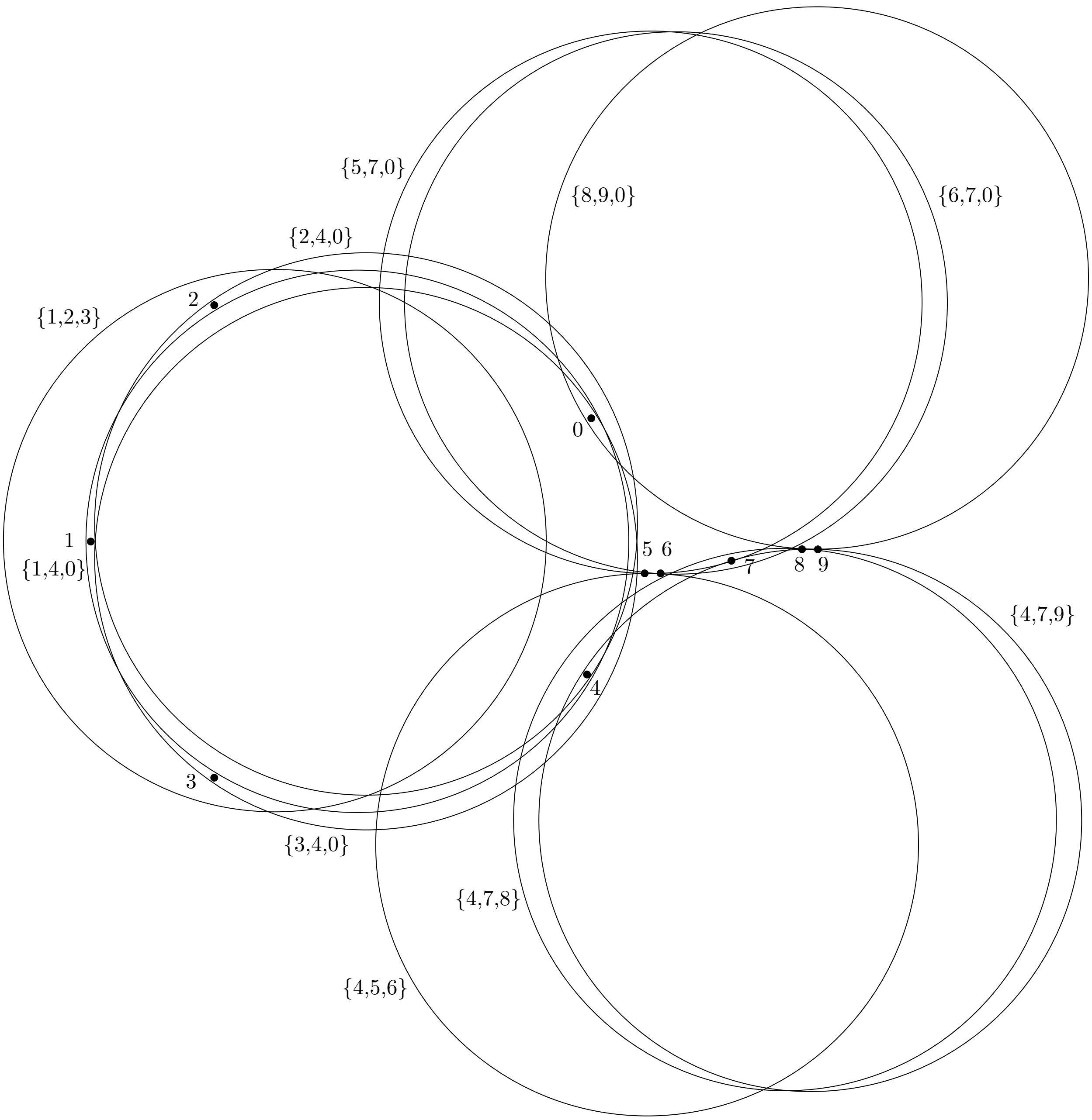


Figure 2: 10 points such that every two-coloring gives disc with 3 points of same color. Numbers next to discs indicate contained points. In the construction obtained by induction for $\mathcal{H}(2,3)$, the point 0 would not be present.

look like two touching discs.) Moreover, also the points of $S(k,l)$ will be no further from each other than ε. Informally, it is also maintained that the segments connecting the centers of discs from different collections are almost vertical and $\mathcal{C}_R$ is “above” $\mathcal{C}_B$. It is easy to see that if $k=1$ or $l=1$, then we have such a collection of unitdisks (see Figure 1(a)).

In the induction step, we place p into the origin, $(0,0)$, place $S(k-1,l)$ nearby $(-\varepsilon/3, -\varepsilon^2/10)$ and $S(k,l-1)$ nearby $(\varepsilon/3, \varepsilon^2/10)$, where $\varepsilon > 0$ is to be determined (see Figure 1(c)). A simple calculation shows that if ε is small enough but much bigger than $\varepsilon(k-1,l)$ and $\varepsilon(k,l-1)$, then this construction indeed realizes $\mathcal{H}$ and satisfies the properties that we required. We omit the details, just prove that for example if $C \in \mathcal{C}_R(k-1,l)$ and $s \in S(k,l-1)$, then $s \notin C$ but $p=(0,0) \in C$. The coordinates of the center of C are $\big(-\varepsilon/3 \pm \varepsilon(k-1,l), 1-\varepsilon^2/10 \pm \varepsilon(k-1,l)\big)$, so the distance of p from C is $(\varepsilon/3)^2 + (1-\varepsilon^2/10)^2 + o(\varepsilon^2) < 1$. On the other hand, the coordinates of s are $\big(\varepsilon/3 \pm \varepsilon(k-1,l), \varepsilon^2/10 \pm \varepsilon(k,l-1)\big)$, thus the square of its distance from the center of C is $(2\varepsilon/3)^2 + (1-2\varepsilon^2/10)^2 + o(\varepsilon^2) > 1$. $\square$

See Figure 2 for an illustration of a version of the construction, which have been modified to ease visibility. It gives 10 points such that in any two-coloring there is a unitdisk that contains exactly three of the points and all of them have the same color.

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