FINAL REPORT **Project K104251**

Classification of finite algebras via compatible relations

In **[13]** we prove that if a variety V (i) has a difference term and (ii) has a finite residual bound, then V is finitely axiomatizable. This results is the first common generalization of two celebrated finite basis theorems by McKenzie [M] and Willard [W] where, instead of (i), V was assumed to be congruence modular or congruence meet-semidistributive, respectively. In **[22]** we show that if in a variety V all finitely generated algebras are free, then V is definitionally equivalent to the variety of sets or pointed sets, or to a variety of vector spaces or affine spaces over a division ring. Both results rely heavily on the study of centrality (commutator, abelianness) for the algebras of V.

'Existence of a cube term' is a recently identified Maltsev condition that promises to be significant; it has found applications in structure theory, natural duality theory, and the study of constraint satisfaction problems. The main result of **[23]** is that an idempotent variety has a d-cube term if and only if its free algebra on two generators has no d-ary compatible cross. It follows that 'existence of a cube term' is join prime in the lattice of idempotent Maltsev conditions.

For any algebra **A**, we define the growth rate function of **A** at n to be the least d such that $\mathbf{A}^n$ has a d-element generating set. The papers **[5]** and **[11]** extend results on finite algebras reported for OTKA T077409 – e.g., a far reaching generalization of Wiegold's theorem (1974) on finite groups saying that **G** has logarithmic growth rate if **G** is perfect, and linear growth rate otherwise – to infinite algebras **A** such that some high enough power of **A** is finitely generated. The third part of the series, **[12]**, is about the growth rates of finite solvable algebras.

In **[29]** we have started the systematic study of compatible quasiorders of algebras in a variety, and shown that many congruence properties – e.g., congruence distributivity [modularity] – satisfied by a locally finite variety carry over to compatible quasiorders. We have also shown that meet semi-distributivity is not preserved: the variety of semilattices is congruence meet semi-distributive but not quasiorder meet semi-distributive. However, from tame congruence theory we know that this property is equivalent to the lack of an M_3 sublattice of congruence lattices, and we prove that this latter property is in fact satisfied by quasiorder lattices of finite algebras in a meet semi-distributive variety.

Two finite algebras are term equivalent if and only if they have the same compatible relations. Categorical equivalence provides a coarser classification: two algebras are categorically equivalent if and only if their compatible relations form isomorphic relational clones [DL]. We investigate categorical equivalence of finite rings with identity in **[9]**. We characterize categorically equivalent pairs of semisimple rings, and we prove that every categorical equivalence between a p-ring and a q-ring is a consequence of the well-known categorical equivalences between finite fields [BB]. It only remains to determine when two p-rings can be categorically equivalent; we conjecture that this can happen only if the rings are term equivalent.

With every digraph we associate an algebra whose fundamental operations are the polymorphisms of the digraph. The digraph of endomorphisms of any finite connected reflexive digraph was proved to be connected in [MZ], provided that the algebra associated with the digraph lies in a variety omitting the Hobby-McKenzie types 1 and 5. A digraph is smooth, if it has no sinks and no sources. Smooth digraphs of algebraic length 1 are a broad generalization of reflexive digraphs. In [BKN], Barto et al. proved that every finite smooth digraph of algebraic length 1, whose associated algebra lies in a variety omitting type 1, has a loop edge. This is a powerful theorem that has nice applications in algebra and computer science. In **[3]** we proved that the digraph of unary polynomial operations of the algebra associated with a finite connected smooth digraph of algebraic length 1 is connected, provided that the algebra lies in a variety omitting types 1 and 5. This generalizes the connectivity result of [MZ] and implies a restricted version of the result of [BKN] mentioned above. We provide an example to show that omitting types 1 and 5 cannot be replaced by omitting type 1 in our theorem.

Dualizability for finite algebras

A finite algebra **A** is called dualizable if it can serve as the character algebra for a 'natural duality', which is a categorical duality similar to Stone duality and Pontryagin duality. In **[10]** we obtain a general sufficient condition for a finite algebra **A** to be dualizable. We prove that if **A** has a cube term and satisfies a condition on congruences which we call the 'split centralizer condition', then **A** is dualizable. This theorem implies all known dualizability results where the algebra has a cube term and generates a residually small variety, and also yields new dualizability results; e.g., the dualizability of finite modules, and the dualizablity of finite *R*-algebras (commutative or not, unital or not) in residually small varieties, where *R* is any commutative unital ring.

Composition of classes of operations

Although the lattice of clones over the two-element set is countable, there is a continuum of partial clones of Boolean functions, and the structure of the lattice of partial clones is largely unknown. The set of partial clones whose total part coincides with a given Boolean clone *C* forms an interval I(*C*) in the partial clone lattice, and Lau posed the problem of determining the cardinalities of these intervals 25 years ago. The cardinality of I(*C*) has been found for several clones since then, and in **[17]** we solve the remaining cases, thereby providing a complete solution to Lau's problem. In **[1]** we investigate more closely an uncountable interval found in [17], and we describe the structure of the bottom and the top of this interval. In **[18]** we study the "middle" of the interval: we prove that there is a continuum of elements below and above each element in this sublattice.

A minor of a function *f* is any function that can by obtained from *f* by identifying variables, permuting variables or adding inessential variables. The minor relation is a quasiorder, and it has been shown in [EFFH] that a class of functions is definable by functional equations if and only if it is an ideal in the minor quasiorder, hence such classes are called equational classes. In **[26]** we consider composition-closed equational classes that are not assumed to contain projections. These classes provide a generalization of clones, and we establish the corresponding generalizations of Rosenberg's theorems on maximal and minimal clones. In **[24]** we characterize the principal ideals of the minor quasiorder in terms of colorings of partition lattices, and we provide infinite families of examples of such "minor posets".

Machida and Rosenberg observed that essentially minimal clones are closely related to so-called lazy operations. An operation *f* is said to be lazy if every operation in the clone generated by *f* is either a projection or a minor of *f*. In **[6]** we describe lazy binary operations, and we obtain new examples of essentially minimal clones as well as a characterization of binary essentially minimal clones for one of the several types considered by Machida and Rosenberg.

The structure of regular and "nearly" regular semigroups

A well-known open problem formulated by Henckell and Rhodes [HR] asks whether each finite inverse monoid has a finite *F*-inverse cover. By Auinger et al. [AS], it is reduced to the question of whether there exists a locally finite group variety for every finite graph such that certain descending chain of subgraphs determined by the variety possesses some property. For a fixed group variety, this property is described in **[14]** by forbidden minors. For abelian varieties, the minors are explicitly given. The approach in [AS] is based on the study of F-inverse covers of Margolis-Meakin expansions of groups. It is generalized in **[15]** to a much larger class of inverse monoids, called E-unitary finite-above inverse monoids. The new graphical criterion obtained is applied to give a simple sufficient condition for an E-unitary finite-above inverse monoid to have no F-inverse cover via the variety of abelian groups.

Under mild conditions on a connected topological space X, its connected covers are known to be classified via conjugacy classes of subgroups of the fundamental group of X. We extend these results in **[7]** to the study of immersions into 2-dimensional CW-complexes. By replacing the fundamental group of a 2-complex C with an appropriate inverse monoid, we classify the immersions over C via conjugacy classes of closed inverse submonoids of this associated monoid. Furthermore, we prove that given a closed inverse submonoid, the corresponding 2-complex can be effectively built.

We study affine representations of polycyclic monoids in **[20]**, which are related to representations of the Cuntz C^*-algebras. We construct infinite families of representations with only one atom, we characterize smooth representations, and we also obtain sharp asymptotic results for the number of atoms.

As a generalization of a well-known embedding theorem for inverse semigroups, we prove in **[19]** that each restriction semigroup is embeddable in an almost factorizable restriction semigroup. As a consequence, we establish that any restriction semigroup has a proper cover which arises from such an embedding and which is a restriction subsemigroup of an inverse semigroup.

In **[2]** an example of an extension of a completely simple semigroup U by a group G is provided which cannot be embedded into the wreath product of U by G. On the other hand, every extension of a completely simple semigroup U by a group G is proved to be embeddable in a semidirect product of a completely simple semigroup V by G where the maximal subgroups of V are direct powers of those of U.

Motivated by Kambites' remark [K] "the lambda-semidirect product is somewhat unusual, indeed arguably even unnatural, in the context of inverse semigroup theory", we introduce a construction called I-semidirect product in **[32]**, which is natural in the context of inverse semigroup theory, and generalizes the usual semidirect product of groups. Moreover, the so-called strong I-semidirect products are proved to be equivalent to lambda-semidirect products from the point of view of which extensions of inverse semigroups are embeddable in them.

The partial automorphisms of a given graph, i.e., the automorphism between its spanned subgraphs, form an inverse monoid. In **[31]** we characterize which inverse monoids of partial bijections are of this form, and describe the inverse monoids isomorphic to the monoid of partial automorphisms of a graph.

Solvability of systems of equations and other computational problems

Let **A** be a finite algebra in a finite language. The subpower membership problem SMP(**A**) is the following combinatorial decision problem: given a list of generators for a compatible relation R of **A** and another tuple b, decide if b belongs to R. If **A** is a group, a ring, or a lattice, then it is known that there exist polynomial time algorithms for solving SMP(**A**). However, it is an open problem whether there is a polynomial time algorithm for SMP(**A**) for all finite algebras **A** with a cube term.

In **[27]** we prove that there exists such an algorithm if **A** generates a residually small variety, or equivalently, if the centralizer of the monolith of every subdirectly irreducible algebra in HS(**A**) is abelian. In **[30]** we prove that the same conclusion holds if the assumption 'abelian' is weakened to 'supernilpotent'.

To help programming a computer to decide whether the variety generated by a finite algebra in a finite language has a (weak) difference term, we produce in **[21]** a simple Maltsev condition which characterizes (weak) difference terms in the class of locally finite varieties.

Equations over algebras are usually studied from the point of view of the complexity of finding a solution. In **[25]** we consider a different aspect: we study the structure of the set of all solutions. We show that the solution set of a system of equations over a finite algebra **A** is always closed under the centralizer of the clone of term operations of **A**, and we prove that over 2-element algebras this condition characterizes solution sets. We also obtain a characterization of solution sets of systems of Boolean functional equations.

Finally, we consider order-theoretic models of preferences in **[8]**: we describe preference relations representable by polynomial functions over distributive lattices. In **[4]** and **[28]** we consider the corresponding lattice polynomial interpolation problem; we present an algorithm to solve this problem, and we prove that the problem is NP-complete if the number of variables is at least 4, and it is in P otherwise.

References

[AS] Auinger K., Szendrei M. B., On F -inverse covers of inverse monoids, J. Pure Appl. Algebra 204: 493-506, 2006.

[BKN] Barto L., Kozik M., Niven T., The CSP dichotomy holds for digraphs with no sources and no sinks (a positive answer to a conjecture of Bang-Jensen and Hell), SIAM Journal on Computing 38 (5): 1782-1802, 2009.

[BB] Bergman C., Berman J., Morita equivalence of almost-primal clones, J. Pure Appl. Algebra 108: 175-201, 1996.

[DL] Denecke K., Lüders O., Categorical equivalence of varieties and invariant relations, Algebra Universalis 46: 105-118, 2001.

[EFHH] Ekin O., Földes S., Hammer P. L., Hellerstein L., Equational characterizations of Boolean function classes, Discrete Math. 211: 27-51, 2000.

[HR] Henckell K., Rhodes J., The theorem of Knast, the PG = BG and type II conjectures, Monoids and Semigroups with Applications (Berkeley, CA, 1989), World Scientific, River Edge, pp. 453-463, 1991.

[K] Kambites M., Anisimov's Theorem for inverse semigroups, Internat. J. Algebra Comput. 26: 41-49, 2015.

[MZ] Maróti M., Zádori L., Reflexive digraphs with near unanimity polymorphisms, Discrete Mathematics 12 (15): 2316-2328, 2012.

[M] McKenzie R., Finite equational bases for congruence modular varieties, Algebra Universalis 24: 224-250, 1987.

[W] Willard R., A finite basis theorem for residually finite, congruence meet-semidistributive varieties, J. Symbolic Logic 65: 187-200, 2000.

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