Harmonic analysis and applications

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PUBLICATIONS

[1] A. G. Babenko, M. V. Deikalova, and Sz. Gy. Révész, *Weighted one-sided approximation of characteristic functions of intervals by polynomials on a closed interval*, P. Steklov Inst. Math. (Supplementary Issues) **21** (2015), no. 4, 46–53.

[2] A. de Roton and Sz. Gy. Révész, *Generalization of the effective Wiener–Ikehara theorem*, Int. J. Number Theory **9** (2013), no. 8, 2091–2128.

[3] T. Eisner, B. Farkas, M. Haase, and R. Nagel, *Operator theoretic aspects of ergodic theory*, Graduate Text in Mathematics, vol. 272, Springer-Verlag, 2015.

[4] B. Farkas, *A note on the periodic decomposition problem for semigroups*, Semigroup Forum (2015), online first, doi: 10.1007/s00233-015-9719-z.

[5] B. Farkas, *Wiener's lemma and the Jacobs–de Leeuw–Glicksberg decomposition*, Ann. Univ. Sci. Budapest. Eötvös Sect. Math. **58** (2015), 25–26, Sebestyén 70, Festschrift.

[6] B. Farkas, *A Bohl–Bohr–Kadets type theorem characterizing Banach spaces not containing c_0*, (2013), arXiv:1301.6250.

[7] B. Farkas, B. Nagy, and Sz. Gy. Révész, *A potential theoretic minimax problem on the torus*, (2015), arXiv:1512.09169.

[8] B. Farkas, J. Pintz, and Sz. Gy. Révész, *On the optimal weight function in the Goldston–Pintz–Yildirim method for finding small gaps between consecutive primes*, Paul Turán Memorial Volume: Number Theory, Analysis and Combinatorics (Berlin), de Gruyter, 2013, pp. 75–104.

[9] B. Farkas and Sz. Gy. Révész, *The periodic decomposition problem*, Springer Proc. in Math. and Stat.: 19th Int. Conf. on Difference Equations and Appl., ICDEA 2013 (Muscat, Oman, 2013), 2014, pp. 143–169.

[10] B. Farkas and S.-A. Wegner, *Variations on Barbalat's lemma*, Amer. Math. Monthly (2016), to appear.

[11] P. Y. Glazyrina and Sz. Gy. Révész, *Turán type oscillation inequalities in L^q norm on the boundary of convex domains*, (2015), arXiv:1512.08268.

[12] A. P. Horváth, *Weighted Fejér constants and Fekete sets*, Acta Math. Hungar. **141** (2013), no. 4, 366–382.

[13] A. P. Horváth, *Müntz-type theorems on the half-line with weights*, J. Math. Anal. Appl. **410** (2014), 699–712.

[14] A. P. Horváth, *The electrostatic properties of zeros of exceptional Laguerre and Jacobi polynomials and stable interpolation*, J. of Approx. Theory **194** (2015), 87–107.

[15] A. P. Horváth, *p-transfinite diameter and p-Chebyshev constant in locally compact space*, Ann. Acad. Sci. Fenn. Math. **40** (2015), 851–874.

[16] A. P. Horváth, *Chromatic derivatives and expansions with weights*, (2016), arxiv:1601.06135, submitted.

[17] A. P. Horváth, *The energy function with respect to the zeros of the exceptional Hermite polynomials*, (2016), arxiv:1601.06138, submitted.

[18] A. P. Horváth and P. Vértesi, *A contribution to the Grünwald–Marcinkiewicz theorem*, Jaen J. on Approx. **4** (2012), no. 1, 1–13.

[19] A. P. Horváth and P. Vértesi, *On barycentric interpolation II. (Grünwald–Marcinkiewicz type theorems)*, Acta Math. Hungar. **148** (2016), no. 1, 147–156.

[20] S. Krenedits, *Three-term idempotent counterexamples in the Hardy–Littlewood majorant problem*, J. Math. Anal. Appl. **388** (2012), no. 1, 136–150.

[21] S. Krenedits, *On Mockenhoupt's conjecture in the Hardy–Littlewood majorant problem*, Journal of Contemporary Mathematical Analysis (Izv. Arm. Acad. Nauk) **48** (2013), no. 3, 91–109.

[22] S. Krenedits, *Fourier analysis — Extremal problems*, Ph.D. thesis, Budapesti Műszaki és Gazdaságtudományi Egyetem Természettudományi Kar, 2014.

[23] S. Krenedits, *Special quadrature error estimates and their application in the Hardy–Littlewood majorant problem*, Acta Mathematica Academiae Paedagogicae Nyíregyháziensis **28** (2912), no. 2, 121–151.

[24] S. Krenedits and Sz. Gy. Révész, *The Carathéodory–Fejér type extremal problem on locally compact abelian groups*, J. Approx. Theory **194** (2015), 108–131.

[25] S. Krenedits and Sz. Gy. Révész, *The point value maximization problem for positive definite functions supported in a given subset of a locally compact group*, (2015), arXiv:1504.03808.

[26] Sz. Gy. Révész, *Turán–Erőd type converse Markov inequalities for convex domains on the plane*, Proceedings of the International Conference "Complex Function Theory and Applications '13", (held in Sofia, Bulgaria, 2013) (V. Kiryakova, ed.), 2013, electronic.

[27] Sz. Gy. Révész, *A discrete extension of the Blaschke rolling ball theorem*, Geometriae Dedicata (to appear).

1. Detailed account of research achievements by the participants

[1] A. G. Babenko, M. V. Deikalova, and Sz. Gy. Révész, *Weighted one-sided approximation of characteristic functions of intervals by polynomials on a closed interval*, P. Steklov Inst. Math. (Supplementary Issues) **21** (2015), no. 4, 46–53.

We have considered the problem of weighted one-sided approximation in L^1 norm on the interval $(-1,1]$ of characteristic functions of intervals $(a,1] \subset (-1,1)$ and $(a,b) \subset (-1,1)$ by algebraic polynomials. This question is of importance in particular for optimal constants in large sieve type estimates as well as in other problems of analytic and combinatorial nature. In the case of half-open intervals, the problem has been solved completely. We have also constructed an example to illustrate the difficulties arising in the case of an open interval.

[2] A. de Roton and Sz. Gy. Révész, *Generalization of the effective Wiener–Ikehara theorem*, Int. J. Number Theory **9** (2013), no. 8, 2091–2128.

We have considered the classical Wiener–Ikehara Tauberian theorem, with the generalized condition of slow decrease and some additional poles on the boundary of convergence of the Laplace transform. In this generality, we have proved the otherwise known asymptotic evaluation of the transformed function, when the usual conditions of the Wiener–Ikehara theorem hold. However, our version also provides an effective error term, seemingly not derived thus far for this general case. The crux of the proof is a proper variation of the lemmas of Ganelius and Tenenbaum, also constructed for the sake of an effective version of the Wiener–Ikehara Theorem.

[3] T. Eisner, B. Farkas, M. Haase, and R. Nagel, *Operator theoretic aspects of ergodic theory*, Graduate Text in Mathematics, vol. 272, Springer-Verlag, 2015.

The book contains results of 8 years of research and work, and finally appeared in 2015 at Springer-Verlag. Not only classical results from operator theory and ergodic theory have been treated in the book, but also new results have been included, while a great emphasis is put on elegant and self-contained proofs. Both ergodic theory and operator theory are strongly connected to harmonic analysis, and the topic of the book relates at several places to the present project. These connections are also highlighted in many places in the book, we just mention the periodic decomposition problem and mean ergodic operators. The distinguishing features of the book as regards harmonic analysis include: a self-contained treatment of compact groups and their Banach space representations, a detailed presentation of the splitting theory of Jacobs–de Leeuw–Glicksberg, Fourier analysis on compact groups, almost periodic functions an the Bohr compactification, the spectral theorem, an operator theoretic version of Wiener's lemma about continuous measures, Rajchman measures, the theorem of Bochner–Herglotz about positive definite functions, Gowers–Host–Kra uniformity seminorms.
Below is a more detailed but nevertheless short synopsis of the book.

Chapter 1 entitled "What is Ergodic Theory?" contains a brief and intuitive introduction to the subject, including some remarks on its historical development. The mathematical theory then starts in Chapters 2 and 3 with *topological dynamical systems*. There, we introduce the basic notions (transitivity, minimality and recurrence) and cover the standard examples, constructions and results like, for instance, the Birkhoff recurrence theorem.

Operator theory appears first in Chapter 4 when we introduce, the *Koopman operator* T on the Banach space $C(K)$ induced by a topological dynamical system $(K;\varphi)$. After providing some classical results on spaces $C(K)$ (Urysohn's lemma, theorems of Tietze and Stone–Weierstraß) we emphasize the Banach algebra structure and give a proof of the classical Gelfand–Naimark theorem. This famous theorem allows to represent each commutative C^*-algebra as a space $C(K)$ and leads to an identification of Koopman operators as the morphisms of such algebras.

In Chapter 5 we introduce *measure-preserving dynamical systems* and cover standard examples and constructions. In particular, we discuss the correspondence of measures on a compact space K with bounded linear functionals on the Banach space $C(K)$. (The proof of the central result here, the Riesz representation theorem, is deferred to Appendix E.) The classical topics of recurrence and ergodicity as the most basic properties of measure-preserving systems are discussed in Chapter 6.

Subsequently, in Chapter 7, we turn to the corresponding operator theory. As in the topological case, a measure-preserving map φ on the probability space X induces a *Koopman operator* T on each space $L^p(X)$. While in the topological situation we look at the space $C(K)$ as a Banach algebra and at the Koopman operator as an algebra homomorphism, in the measure theoretic context the corresponding spaces are *Banach lattices* and the Koopman operators are *lattice homomorphisms*. Consequently, we include a short introduction into abstract Banach lattices and their morphisms. Finally, we characterize the ergodicity of a measure-preserving dynamical system by the fixed space or, alternatively, by the irreducibility of the Koopman operator.

After these preparations, we discuss the most central operator theoretic results in ergodic theory, von Neumann's *mean ergodic theorem* (Chapter 8) and Birkhoff's *pointwise ergodic theorem* (Chapter 11). The former is placed in the more general context of *mean ergodic operators* and in Chapter 10 we discuss this concept for Koopman operators of topological dynamical systems. Here, the classical results of Krylov–Bogoljubov about the existence of invariant measures is proved and the concepts of *unique* and *strict ergodicity* are introduced and exemplified with Furstenberg's theorem on group extensions.

In between the discussion of the ergodic theorems, in Chapter 9, we introduce the concepts of strongly and weakly *mixing* systems. This topic has again a strong operator theoretic flavor, as the different types of mixing are characterized by different asymptotic behavior of the powers T^n of the Koopman operator as $n \to \infty$. The full picture of weakly-mixing systems is eventually revealed in Chapter 16, when this compactness is studied in detail (see below).

Next, in Chapter 12, we consider different concepts of "isomorphism"—point isomorphism, measure algebra isomorphism and Markov isomorphism—of measure-preserving systems. From a classical point of view, the notion of point isomorphism appears to be the most natural. In our view, however, the Koopman operators contain all essential information of the dynamical system and underlying state space maps are secondary. Therefore, it becomes natural to embed the class of "concrete" measure-preserving systems into the larger class of "abstract" measure-preserving systems and use the corresponding notion of (Markov) isomorphism. By virtue of the Gelfand–Naimark theorem, each abstract measure-preserving system has many concrete *topological models*. One canon-ical model, the *Stone representation* is discussed in detail.

In Chapter 13 we introduce the class of *Markov operators*, which plays a central role in later chapters. Different types of Markov operators (embeddings, factor maps, Markov projections) are discussed and the related concept of a *factor* of a measure-preserving system is introduced.

Compact groups feature prominently as one of the most fundamental examples of dynamical systems. A short yet self-contained introduction to their theory is the topic of Chapter 14. For a better understanding of dynamical systems, we present the essentials of Pontryagin's duality theory for compact/discrete Abelian groups. This chapter is accompanied by the results in Appendix G, where the existence of the Haar measure and Ellis' theorem for compact semitopological groups is proved in its full generality. In Chapter 15 we discuss group actions and linear representations of compact groups on Banach spaces, with a special focus on representations by Markov operators.

In Chapter 16 we start with the study of compact *semigroups*. Then we develop a powerful tool for the study of the asymptotic behavior of semigroup actions on Banach spaces, the *Jacobs–de Leeuw–Glicksberg* (JdLG-)decomposition. Applied to the semigroup generated by a Markov operator T it yields an orthogonal splitting of the corresponding L^2-space into its "reversible" and the "almost weakly stable" part. The former is the range of a Markov projection and hence a factor, and the operator generates a compact group on it. The latter is characterized by a convergence to 0 (in some sense) of the powers of T.

Applied to the Koopman operator of a measure-preserving system, the reversible part in the JdLG-decomposition is the so-called *Kronecker factor*. It turns out that this factor is trivial if and only if the system is weakly mixing. On the other hand, this factor is the whole system if and only if the Koopman operator has *discrete spectrum*, in which case the system is (Markov) isomorphic to a rotation on a compact monothetic group (Halmos–von Neumann theorem, Chapter 17).

Chapter 18 is devoted to the spectral theory of dynamical systems. Based on a detailed proof of the spectral theorem for normal operators on Hilbert spaces, the concepts of maximal spectral type and spectral multiplicity function are introduced. The chapter concludes with a series of instructive examples.

In Chapter 19 we approach the Stone–Čech compactification of a (discrete) semigroup via the Gelfand–Naimark theorem and return to topological dynamics by showing some less classical results, like the theorem of Furstenberg and Weiss about multiple recurrence. Here we encounter the first applications of dynamical systems to combinatorics and prove the theorems of van der Waerden, Gallai and Hindman.

In Chapter 20 we describe Furstenberg's correspondence principle, which establishes a relations between ergodic theory and combinatorial number theory. As an application of the JdLG-decomposition we prove the existence of arithmetic progressions of length 3 in certain subsets of $\mathbb{N}$, i.e., the first non-trivial case of Szemerédi's theorem on arithmetic progressions. We also indicate briefly how the general Szemerédi–Furstenberg theorem can be proved via Gowers–Host–Kra uniformity norms.

Finally, in Chapter 21, more ergodic theorems lead the reader to less classical areas and to the front of active research.

[4] B. Farkas, *A note on the periodic decomposition problem for semigroups*, Semigroup Forum (2015), online first, doi: 10.1007/s00233-015-9719-z.

Given $T_1, \dots, T_n$ commuting power-bounded operators on a Banach space we have studied under which conditions the equality $\ker(T_1-I)\cdots(T_n-I)=\ker(T_1-I)+\cdots+\ker(T_n-I)$ holds true. This problem, known as the periodic decomposition problem, goes back to I. Z. Ruzsa. In this short note we have considered the case when $T_j=T(t_j)$, $t_j>0$, $j=1,\dots,n$ for some one-parameter semigroup $(T(t))_{t\geq 0}$. We also have looked at a generalization of the periodic decomposition problem when instead of the cyclic semigroups $\{T_j^n : n\in\mathbb{N}\}$ more general semigroups of bounded linear operators are considered.

[5] B. Farkas, *Wiener's lemma and the Jacobs–de Leeuw–Glicksberg decomposition*, Ann. Univ. Sci. Budapest. Eötvös Sect. Math. **58** (2015), 25–26, Sebestyén 70, Festschrift.

A result of J.A. Goldstein extends Wiener's lemma from harmonic analysis about continuous and atomic measures to Hilbert space contractions. In this short note, we have discussed the relation between that lemma of Wiener and the Jacobs–de Leeuw–Glicksberg decomposition from operator theory via spectral measure. By using this decomposition we have proved Goldstein's result in a way that is closer to the elementary proof of Wiener's lemma, and in a slightly stronger form at that.

[6] B. Farkas, *A Bohl–Bohr–Kadets type theorem characterizing Banach spaces not containing* c_0, (2013), arXiv:1301.6250.

We have proved that a separable Banach space E does not contain a copy of the space c_0 of null-sequences if and only if for every doubly power-bounded operator T on E and for every vector $x\in E$ the relative compactness of the sets $\{T^{n+m}x-T^nx : n\in\mathbb{N}\}$ (for some/all $m\in\mathbb{N}$, $m\geq 1$) and $\{T^nx : n\in\mathbb{N}\}$ are equivalent. With the help of the Jacobs–de Leeuw–Glicksberg decomposition of strongly compact semigroups the case of (not necessarily invertible) power-bounded operators has been also handled. These results are analogues of the Bohr–Bohl–Kadets theorem about integration of vector-valued almost periodic functions.

[7] B. Farkas, B. Nagy, and Sz. Gy. Révész, *A potential theoretic minimax problem on the torus*, (2015), arXiv:1512.09169.

We have investigated an extension of an equilibrium-type result, conjectured by Ambrus, Ball and Erdélyi, and proved recently by Hardin, Kendall and Saff. These results were formulated on the torus, hence we have also worked on the torus, but one of the main motivations for our extension comes from an analogous setup on the unit interval, investigated earlier by Fenton.

Basically, the problem is a minimax one, i.e., to minimize the maximum of a function F, defined as the sum of arbitrary translates of certain fixed "kernel functions", minimization understood with respect to the translates. If these kernels are assumed to be concave, having certain singularities or cusps at zero, then translates by y_j will have singularities at y_j (while in between these nodes the sum function still behaves relatively regularly). So one can consider the maxima m_i on each subintervals between the nodes y_j, and look for the minimization of $\max F=\max_i m_i$.

Here also a dual question of maximization of $\min_i m_i$ arises. This type of minimax problems were treated under some additional assumptions on the kernels. Also the problem is normalized so that $y_0=0$.

In particular, Hardin, Kendall and Saff assumed that we have one single kernel K on the torus or circle, and $F=\sum_{j=0}^n K(\cdot-y_j)=K+\sum_{j=1}^n K(\cdot-y_j)$. Fenton considered situations on the interval with *two* fixed kernels J and K, also satisfying additional assumptions, and $F=J+\sum_{j=1}^n K(\cdot-y_j)$. Here we have considered the situation (on the circle) when *all the kernel functions can be different*, and $F=\sum_{j=0}^n K_j(\cdot-y_j)=K_0+\sum_{j=1}^n K_j(\cdot-y_j)$. Also an emphasis is put on relaxing all other technical assumptions and give alternative, rather minimal variants of the set of conditions on the kernel.

[8] B. Farkas, J. Pintz, and Sz. Gy. Révész, *On the optimal weight function in the Goldston–Pintz–Yildirim method for finding small gaps between consecutive primes*, Paul Turán Memorial Volume: Number Theory, Analysis and Combinatorics (Berlin), de Gruyter, 2013, pp. 75–104.

We have analyzed the choice of the weight function in the Goldston–Pintz–Yildirim method for finding small gaps between primes. Our starting point was the optimization problem, initiated by the survey article of Soundararajan, for the choice of some underlying polynomial P used in the construction of the weight function in the said GPY method. With some reformulations we have obtained a maximization problem in a Hilbert space, which can be explicitly solved, moreover, it has turned out that even in this larger space of functions extremals are smooth. The arising differential equations are transformed versions of the Bessel differential equations, thus leading to transformed Bessel functions as solutions and hence as extremals. The analysis gives that the maximal value of the investigated quantity is $4/(k+ck^{1/3})$ and shows that the optimal choice is the $k-1^{\text{st}}$ integral of $u(x):=x^{1-k/2}J_{k-2}(\alpha_1\sqrt{x})$, where $\alpha_1\sim ck^{1/3}$ is the first positive root of the Bessel function J_{k-2}. As this naturally gives rise to a number of technical problems in the application of the GPY method, we also have constructed a polynomial p which is a simpler function yet it furnishes an approximately optimal extremal quantity, $4/(k+Ck^{1/3})$ with some other constant C. In a succeeding paper of J. Pintz it is indeed shown how this quasi-optimal choice of the weight finally can exploit the GPY method to its theoretical limits.

[9] B. Farkas and Sz. Gy. Révész, *The periodic decomposition problem*, Springer Proc. in Math. and Stat.: 19th Int. Conf. on Difference Equations and Appl., ICDEA 2013 (Muscat, Oman, 2013), 2014, pp. 143–169.

If a function $f : \mathbb{R} \to \mathbb{R}$ can be represented as the sum of n periodic functions as $f = f_1 + \cdots + f_n$ with $f(x+\alpha_j) = f(x)$ $(j = 1,\dots,n)$, then it also satisfies a corresponding n^{th} order difference equation $\Delta_{\alpha_1} \dots \Delta_{\alpha_n} f = 0$. The periodic decomposition problem asks for the converse implication, which may hold or fail depending on the context (on the system of periods, on the function class in which the problem is considered, etc.). The problem has natural extensions and ramifications in various directions, and is related to several other problems in real analysis, Fourier and functional analysis. We have given a survey about the available methods and results, and presented a number of intriguing open problems. We have given only some selected proofs, including some alternative new, ones which have not been published, that give substantial insight into the subject matter, or that reveal connections to other mathematical areas.

[10] B. Farkas and S.-A. Wegner, *Variations on Barbalat's lemma*, Amer. Math. Monthly (2016), to appear.

It is not hard to prove that a uniformly continuous real function, whose integral up to infinity exists, vanishes at infinity, and it is probably little known that this statement runs under the name “Barbălat's Lemma”. In fact, the latter name is frequently used in control theory, where the lemma is used to obtain Lyapunov-like stability theorems for non-linear and non-autonomous systems. Barbălat's Lemma is *qualitative* in the sense that it asserts that a function has certain properties (convergence to zero). Such qualitative statements can typically be proved by *soft analysis*, such as indirect proofs. Indeed, in the original 1959 paper by Barbălat, the lemma was proved by contradiction and this proof prevails in the control theory textbooks. In this short note we first have given a direct, *hard analysis* proof of the lemma, yielding *quantitative* results, i.e., *rates* of convergence to zero. This proof allows also for immediate generalizations. We also have unified three different versions which that appeared recently and discussed their relation to the original lemma.

[11] P. Y. Glazyrina and Sz. Gy. Révész, *Turán type oscillation inequalities in L^q norm on the boundary of convex domains*, (2015), arXiv:1512.08268.

Some 77 years ago P. Turán was the first to establish lower estimations of the ratio of the maximum norm of the derivatives of polynomials and the maximum norm of the polynomials themselves on the interval $\mathbb{I} := [-1,1]$ and on the unit disk $\mathbb{D} := \{z \in \mathbb{C} \,:\, |z| \le 1\}$ under the normalization condition that the zeroes of the polynomial all lie in the interval or in the disk, respectively. He proved that with $n := \deg p$ tending to infinity, the precise growth order of the minimal possible ratio of the derivative norm and the norm is $\sqrt{n}$ for $\mathbb{I}$ and n for $\mathbb{D}$.
J. Erőd continued the work of Turán and extended his results to several other domains. The growth of the minimal possible ratio of the ∞-norm of the derivative and the polynomial itself was proved to be of order n for all compact convex domains a decade ago.
Although Turán himself gave comments about the above oscillation question in L^q norms, till recently results were known only for $\mathbb{D}$ and $\mathbb{I}$. Here we have proved that in L^q norm the oscillation order is again n for a certain class of convex domains, including all smooth convex domains and also convex polygonal domains having no acute angles at their vertices.

[12] A. P. Horváth, *Weighted Fejér constants and Fekete sets*, Acta Math. Hungar. **141** (2013), no. 4, 366–382.

In this paper we have indicated the connections between the Fekete sets, the zeros of orthogonal polynomials, $1(w)$-normal point systems, and the nodes of an interpolatory process which is called stable and the most economical, via the Fejér constants. Finally, the convergence of a weighted Grünwald interpolation has been proved.

[13] A. P. Horváth, *Müntz-type theorems on the half-line with weights*, J. Math. Anal. Appl. **410** (2014), 699–712.

We have considered the linear span S of the functions t^{a_k} (with some $a_k > 0$) in weighted L^2 spaces, with rather general weights. We have given one necessary and one sufficient condition for S to be dense. Some comparisons have been also made between the new results and those that can be deduced from older ones in the literature.

[14] A. P. Horváth, *The electrostatic properties of zeros of exceptional Laguerre and Jacobi polynomials and stable interpolation*, J. of Approx. Theory **194** (2015), 87–107.

We have examined the electrostatic properties of exceptional and regular zeros of X_m-Laguerre and X_m-Jacobi polynomials. Since there is a close connection between the electrostatic properties of the zeros and the stability of interpolation on the system of zeros, we could deduce an Egerváry–Turán type result as well. The limit of the energy on the regular zeros has been also investigated.

[15] A. P. Horváth, *p-transfinite diameter and p-Chebyshev constant in locally compact space*, Ann. Acad. Sci. Fenn. Math. **40** (2015), 851–874.

We have extended the notion of transfinite diameter and Chebyshev constant to p-potential theory in locally compact spaces and study their relations. As in the classical case, it turns out that provided that the kernel satisfies a certain condition, for any compact sets the energy, the Chebyshev constant and the transfinite diameter are coincide. The investigations follow the linear method developed by e.g. Choquet, Fuglede, Ohtsuka, Farkas and Nagy. Taking into consideration the significance of finite sets of the minimal and almost minimal energy, we have studied Fekete and greedy energy sets as well.

[16] A. P. Horváth, *Chromatic derivatives and expansions with weights*, (2016), arxiv:1601.06135, submitted.

Chromatic derivatives and series expansions of bandlimited functions have recently been introduced in signal processing and they have been shown to be useful in practical applications. We have extended the notion of chromatic derivative using varying weights. When the kernel function of the integral operator is positive, this extension ensures chromatic expansions around every points. Besides old examples, the modified method has been demonstrated via some new ones as Walsh–Fourier transform, and Poisson-wavelet transform. Moreover, the chromatic expansion of a function in some L^p-space has been investigated.

[17] A. P. Horváth, *The energy function with respect to the zeros of the exceptional Hermite polynomials*, (2016), arxiv:1601.06138, submitted.

We have examined the energy function with respect to the zeros of exceptional Hermite polynomials. The localization of the eigenvalues of the Hessian has been given in the general case. In some special arrangements we have obtained a more precise result on the behavior of the energy function. Finally, we have investigated the energy function with respect to the regular zeros of the exceptional Hermite polynomials.

[18] A. P. Horváth and P. Vértesi, *A contribution to the Grünwald–Marcinkiewicz theorem*, Jaen J. on Approx. **4** (2012), no. 1, 1–13.

This paper is a generalization of the Grünwald–Marcinkiewicz theorem revealing its connection to a process defined by S. N. Bernstein.

[19] A. P. Horváth and P. Vértesi, *On barycentric interpolation II. (Grünwald–Marcinkiewicz type theorems)*, Acta Math. Hungar. **148** (2016), no. 1, 147–156.

The paper proves Grünwald–Marcinkiewicz type theorems with respect to barycentric Lagrange interpolation based on equidistant and Chebyshev node-sytems in $[-1,1]$. It has turned out that the results are very similar to the ones known for the classical Lagrange interpolation.

[20] S. Krenedits, *Three-term idempotent counterexamples in the Hardy–Littlewood majorant problem*, J. Math. Anal. Appl. **388** (2012), no. 1, 136–150.

The Hardy–Littlewood majorant problem was raised in the 30's and it can be formulated as the question whether $\int |f|^p \geq \int |g|^p$ whenever $\widehat{f} \geq |\widehat{g}|$. It has a positive answer only for exponents p which are even integers. Montgomery conjectured that even among the idempotent polynomials there must exist some counterexamples, i.e., there exists some finite set of exponentials and some $\pm$ signs with which the signed exponential sum has larger p^{th} norm than the idempotent obtained with all the signs chosen $+$ in the exponential sum. That conjecture was proved recently by Mockenhaupt and Schlag.
However, a natural question is if even the classical

$$1 + \mathrm{e}^{2\pi i x} \pm \mathrm{e}^{2\pi i (k+2)x} \tag{1}$$

three-term exponential sums, used for $p = 3$ and $k = 1$ already by Hardy and Littlewood, should work in this respect. That remained unproven, since the construction of Mockenhaupt and Schlag works with four-term idempotents. We investigate the sharpened question and show that at least in certain cases there indeed exist three-term idempotent counterexamples in the Hardy–Littlewood majorant problem; that is we have for $0 < p < 6, p \notin 2\mathbb{N}$

$$\int_0^{\frac{1}{2}} |1 + \mathrm{e}^{2\pi i x} - \mathrm{e}^{2\pi i ([\frac{p}{2}]+2)x}|^p \mathrm{d}x > \int_0^{1/2} |1 + \mathrm{e}^{2\pi i x} + \mathrm{e}^{2\pi i ([\frac{p}{2}]+2)x}|^p \mathrm{d}x.$$

The proof combines delicate calculus with numerical integration and precise error estimates.

[21] S. Krenedits, *On Mockenhoupt's conjecture in the Hardy–Littlewood majorant problem*, Journal of Contemporary Mathematical Analysis (Izv. Arm. Acad. Nauk) **48** (2013), no. 3, 91–109.

In the previous work on the Hardy–Littlewood majorant problem, S.K. proved Mockenhaupt's conjecture for $k = 0,1,2$, i.e., in the range $0 < p < 6$, $p \notin 2\mathbb{N}$. Continuing this work here it is demonstrated that even the cases

$k = 3, 4$ hold true. Several refinements in the technical features of the approach include improved fourth order quadrature formulae, use of variation and integral mean estimates in error bounds for the quadrature, and detailed error estimates of approximations of various derivatives in various subintervals, the exact choice of which are governed by global pictures of behavior gained from numerical calculations.

[22] S. Krenedits, *Fourier analysis — Extremal problems*, Ph.D. thesis, Budapesti Műszaki és Gazdaságtudományi Egyetem Természettudományi Kar, 2014.

This thesis, submitted in 2014 and defended in 2015, is based on the papers [20], [21], [23], [24], [25].
The thesis deals with two problems from the area of Fourier analysis. In the first chapter idempotent polynomial counterexamples are given for the Hardy–Littlewood majorant problem for $0 < p < 12$, p not an even integer. Mockenhaupt conjectured, that the classical three-term character sums $1 + e^{2\pi ix} \pm e^{2\pi i(k+2)x}$ are counterexamples for $2k < p < 2k+2$. In the thesis this conjecture is proved for $k = 0, 1, 2, 3, 4, 5$.
It is well-known that the Hardy–Littlewood majorant problem has a positive answer only for exponents p which are even integers, and for $p = \infty$. Montgomery conjectured that even among the idempotent polynomials there must exist some counterexamples, i.e., there exists some finite set of characters and some $\pm$ signs with which the signed character sum has larger p^{th} norm (quasi-norm, if $0 < p < 1$) than the idempotent obtained with all the signs chosen $+$ in the character sum. That conjecture was proved in 2006 by Mockenhaupt and Schlag with an explicit construction of a four term counterexample.
However, already in 1996 Mockenhaupt in his habilitation thesis conjectured that even the classical three-term character sums $1 + e^{2\pi ix} \pm e^{2\pi i(k+2)x}$, used for $p = 3$ and $k = 1$ already by Hardy and Littlewood, should work in this respect for $2k < p < 2k+2$. That remained open.
S. Krenedits investigated this sharpened question and showed this conjecture for $k = 0, 1, 2, 3, 4, 5$; i.e., in the range $0 < p < 12$, $p \notin 2\mathbb{N}$. In case $k = 0$, i.e., $0 < p < 2$, he provides an analytical solution, based on work of Bonami and Révész, while for the other cases various numerical methods, in particular numerical integral estimation procedures are used.
In the second part–which is joint work with his advisor Szilárd Révész–they have considered the extremal problem of maximizing the function value $|f(z)|$ at a given point $z \in G$ attained by some positive definite and continuous function f on a locally compact group G, where for a given symmetric open set $\Omega \ni z$, f vanishes outside Ω and is normalized by $f(0) = 1$. They have extended the former results for classical groups to arbitrary locally compact Abelian (LCA) groups; then they have established a certain extension (under some conditions) on not necessarily commutative locally compact groups.

[23] S. Krenedits, *Special quadrature error estimates and their application in the Hardy–Littlewood majorant problem*, Acta Mathematica Academiae Paedagogicae Nyíregyháziensis **28** (2912), no. 2, 121–151.

In two previous papers we proved this Mockenhaupt's conjecture for $k = 0, 1, 2, 3, 4$, i.e., in the range $0 < p < 10, p \notin 2\mathbb{N}$. In this paper, we have demonstrated that even the $k = 5$ case holds true. Refinements in the technical features of our approach include use of total variation and integral mean estimates in error bounds for a certain fourth order quadrature. Our estimates make good use of the special forms of functions we encounter: linear combinations of powers and powers of logarithms of absolute value squares of trigonometric polynomials of given degree. Thus the quadrature error estimates are less general, but we could get better constants which are of practical use for us.

[24] S. Krenedits and Sz. Gy. Révész, *The Carathéodory–Fejér type extremal problem on locally compact abelian groups*, J. Approx. Theory **194** (2015), 108–131.

We have considered the extremal problem of maximizing a point value $|f(z)|$ at a given point $z \in G$ by some positive definite and continuous function f on a locally compact Abelian group (LCA group) G, where for a given symmetric open set $\Omega \ni z$, f vanishes outside Ω and is normalized by $f(0) = 1$.
This extremal problem was investigated in $\mathbb{R}$ and $\mathbb{R}^d$ and for Ω a 0-symmetric convex body in a paper of Boas and Kac in 1945. Arestov, Berdysheva and Berens extended the investigation to $\mathbb{T}^d$, where $\mathbb{T} := \mathbb{R}/\mathbb{Z}$. Kolountzakis and Révész gave a more general setting, considering arbitrary open sets, in all the classical groups above. Also they observed, that such extremal problems occurred in certain special cases and in a different, but equivalent formulation already a century ago in the work of Carathéodory and Fejér.
Moreover, following observations of Boas and Kac, Kolountzakis and Révész showed how the general problem can be reduced to equivalent discrete problems of "Carathéodory–Fejér type" on $\mathbb{Z}$ or $\mathbb{Z}_m := \mathbb{Z}/m\mathbb{Z}$. We have extended their results to arbitrary LCA groups.

[25] S. Krenedits and Sz. Gy. Révész, *The point value maximization problem for positive definite functions supported in a given subset of a locally compact group*, (2015), arXiv:1504.03808.

The century old extremal problem, solved by Carathéodory and Fejér, concerns a non-negative trigonometric polynomial $T(t) = a_0 + \sum_{k=1}^{n} a_k \cos(2\pi kt) + b_k \sin(2\pi kt) \geq 0$, normalized by $a_0 = 1$, and the quantity to be maximized is the coefficient a_1 of $\cos(2\pi t)$. Carathéodory and Fejér found that for any given degree n the maximum is $2\cos(\frac{\pi}{n+2})$.
In the complex exponential form, the coefficient sequence $(c_k) \subset \mathbb{C}$ will be supported in $[-n,n]$ and normalized by $c_0 = 1$. Reformulating, non-negativity of T translates to positive definiteness of the sequence (c_k), and the extremal problem becomes a maximization problem for the value at 1 of a normalized positive definite function $c : \mathbb{Z} \to \mathbb{C}$, supported in $[-n,n]$.
Boas and Katz, Arestov, Berdysheva and Berens, Kolountzakis and Révész and recently Krenedits and Révész investigated the problem in increasing generality, reaching analogous results for all locally compact Abelian groups. We have proved an extension to all the known results in not necessarily commutative locally compact groups.

[26] Sz. Gy. Révész, *Turán–Erőd type converse Markov inequalities for convex domains on the plane*, Proceedings of the International Conference "Complex Function Theory and Applications '13", (held in Sofia, Bulgaria, 2013) (V. Kiryakova, ed.), 2013, electronic.

For a convex domain $K \subset \mathbb{C}$ the well-known general Bernstein–Markov inequality holds: a polynomial p of degree n must have $\|p'\| \leq c(K)n^2\|p\|$. However, for polynomials in general, $\|p'\|$ can be arbitrarily small, compared to $\|p\|$.
Turán investigated the situation under the condition that p have all its zeroes in the convex body K. With this assumption he proved $\|p'\| \geq (n/2)\|p\|$ for the unit disk D and $\|p'\| \geq c\sqrt{n}\|p\|$ for the unit interval $I := [-1,1]$. Levenberg and Poletsky provided general lower estimates of order $\sqrt{n}$, and there were certain classes of domains with order n lower estimates.
We have shown that for *all* compact and convex *domains* K and polynomials p with all their zeroes in K $\|p'\| \geq c(K)n\|p\|$ holds true, while $\|p'\| \leq C(K)n\|p\|$ occurs for arbitrary compact connected sets $K \subset \mathbb{C}$. Moreover, the dependence on width and diameter of the set K has been found up to a constant factor. Note that if K is *not* a domain ($\mathrm{int}K = \emptyset$), then the order is only $\sqrt{n}$.
Erőd observed that in case the boundary of the domain is smooth and the curvature exceeds a constant $\kappa > 0$, then we can get an order n lower estimation with the curvature occurring in the implied constant. Elaborating on this idea several extensions of the result have been given. Again, geometry is in focus, including a new, strong "discrete" version of the classical Blaschke Rolling Ball Theorem.

[27] Sz. Gy. Révész, *A discrete extension of the Blaschke rolling ball theorem*, Geometriae Dedicata (to appear).

The Rolling Ball Theorem asserts that given a convex body $K \subset \mathbb{R}^d$ in Euclidean space and having a C^2-smooth surface ∂K with all principal curvatures not exceeding $c > 0$ at all boundary points, then K necessarily has the property that to each boundary point there exists a ball B_r of radius $r = 1/c$, fully contained in K and touching ∂K at the given boundary point from the inside of K.
We have proved a discrete analogue of the result on the plane. We have considered a certain discrete condition on the curvature, namely that to any boundary points $\mathbf{x},\mathbf{y} \in \partial K$ with $|\mathbf{x}-\mathbf{y}| \leq \tau$, the angle $\varphi(\mathbf{n_x},\mathbf{n_y}) := \arccos\langle \mathbf{n_x},\mathbf{n_y}\rangle$ of any unit outer normals $\mathbf{n_x},\mathbf{n_y}$ at $\mathbf{x}$ and at $\mathbf{y}$, resp., does not exceed a given angle φ. Then we have constructed a corresponding body, $M(\tau,\varphi)$, which is to lie fully within K while containing the given boundary point $\mathbf{x} \in \partial K$.
In dimension $d = 2$, that is, on the plane, M is almost a regular n-gon, and the result allows to recover the precise form of Blaschke's Rolling Ball Theorem in the limit.
Similarly, we have considered the dual type discrete Blaschke theorems ensuring certain circumscribed polygons. In the limit, the discrete theorem enables us to provide a new proof for a strong result of Strantzen assuming only a.e. existence and lower estimations on the curvature.
For $d \geq 3$, directly we could derive only a weaker, quasi-precise form of the discrete inscribed ball theorem, while no space version of the circumscribed ball theorem has been found. However, at least the higher dimensional smooth cases follow already from the plane versions of the smooth theorems, which follow as limiting cases also from our discrete versions.

2. General description of the project execution and the main additions to the content

At the time of writing the project plan, we were planning a period of time when extensive research can be central in the life of the participants. Although the project is affiliated at a research institute, this ideal time condition did never occur. Two of the participants (B.F. and Á.H.) were continuously teaching at universities, one (S.K.) had to undertake various regular jobs like a secondary school teacher or even as a public employee at various (so to say, somewhat artificially created) jobs for otherwise jobless, and the project coordinator (here not dwelling into discussion of personal life circumstances) but spent most semesters teaching at various universities abroad and home.

The usual setup of such research projects focuses on research publications, but we deviated from this frame when undertaking rather long term projects. We are proud of the completion of the more than 600 pages long book [3], which appeared right at the end of the project duration in the prestigious Graduate Texts in Mathematics series of Springer Verlag. Naturally, this one item in our reference list contains the work of countless hours (also of research) by the (co)author B.F. The book is not just a presentation of known results but contains a good number of new and simplified proofs, or new results that were indeed published also separately. Relevant for this project is, e.g., an operator theoretic version of Wiener's result about continuos measures, see [5].

Another long term project was the undertaking of K.S., who, at the age of 57, defended his PhD thesis [22] under the advising of Sz.R. (which is clearly a cooperation within the project). The thesis comprises the results of five papers, written under the guidance (and two of them jointly with) Sz.R., but, as is natural, apart from writing these papers, to write and defend the thesis, together with the exams etc., also needed considerable time and work (also from the advisor, too, all the more so because S.K. was not a regular PhD student, but studying and working independently and without the courses and support of the doctoral school). Nevertheless, this sacrifice—loss of time in some count—was worthy, since S.K., who had no proper job for years, finally achieved a breakthrough by this undertaking and arrived at a settled status with a university job[1]. The case is a very good point of lifelong learning and training for new jobs and professions—a side, but non-negligible aspect of the measurement of usefulness of such project supports.

Given these facts, we are satisfied with the 27 publications which we have published over the four years. The mathematical scientific writings of well over a thousand pages clearly document our work. Nevertheless, as is natural in such long-term projects, we did not exactly get what we were planning for. Also, some new developments draw our attention to new questions, which were not planned for but turned up and were seen more important at the time than the originally set goals.

An important example is our short adventure taken into the world-wide contest towards the (quasi)-twin-prime conjecture. With a leading role of our colleague J. Pintz, some ten years ago the first results showing that the gaps between consecutive primes may be infinitesimally smaller than the average gaps (i.e., $\log n$) appeared. The Goldston–Pintz–Yildirim method was a top breakthrough of contemporary mathematics. To see its importance note that in 2014 Goldstone, Pintz, Yildirim and Zhang were awarded the Cole Prize. Over the next years better and better results appeared and e.g. it was proved that the prime gaps may be smaller than $\sqrt{\log n}$ infinitely often. Soundararajan analyzed the method in the Bulletin of the Amer. Math. Soc., concluding that the key technical parameter, which determines the resulting order of this estimation, is directly connected to a—non-trivial—extremal problem. Then J. Pintz proposed us to deal with this extremal problem, which, on the one hand, was another extremal problem not far from our original research plans and topics, but, on the other hand, clearly was not planned for and set forth. We consider that it was the right decision to work on this problem, solve it completely (B.F. and Sz.R. jointly with J. Pintz) [8], and allow J. Pintz to subsequently push the GPY method to its theoretical limits (achieving gaps of size $\log^{3/7} n$). The solution of Soundararajan's extremal problem did not became more important only because very shortly after our work, Y. Zhang found another breakthrough and proved even constant gaps. However, the subsequent efforts to bring down the constant (Tao's Polymath Projects) still used our results and so the famous polymath paper refers to our work, too. Thus our project contributed to the impressive presence of Hungarian mathematics in this world-sensational decade of progress in the twin prime problem.

A second example is the complex polynomial extremal problems of Turán–Erőd type, which we did not plan for. However, slowly by slowly this topic attracted our attention again. In this circle of problems the question is about the lower estimation of the derivative of polynomials, normalized to have norm one and having all their zeroes in a certain compact, convex set (domain) $K \subset \mathbb{C}$. The problem was investigated first by Turán for the unit disk D

[1]Even if at a relatively old age, S.K., defending his PhD and subsequently finding his proper job—and thus leaving the welfare system for jobless people—spared more money to the state than the whole project funding totals for the whole duration.

and the interval $I := [-1,1]$, then extended by Erőd to further domains, and for over 70 years most results could deal only with the fundamental sets D and I. A decade ago Sz.R. solved the question of the order of oscillation in full generality of all convex compact domains regarding the ∞-norm. Our recent investigations attacked the (more difficult) case of L^q norms. This extremal problem, though similar in nature to the other extremal problems of the project plan, but turned to be more intimately connected to geometry questions than to harmonic analysis. We had even written a purely geometry paper [27] to which our further results are based, while several other papers [26, 11]—also bringing in some international cooperation—were written (and others are in progress, too). Our results are indeed the first ones which may obtain non-trivial—and, in fact, of optimal order!—estimates on oscillation of polynomials, normalized as above, in L^q norm on general domains (and, at all, different form D and I, which were essentially the only cases successfully dealt with so far for L^q norm with $q < \infty$).

3. Realization of concrete research advances as compared to the original project proposal

Coming to the concrete plans, we originally planned to address questions sorted into two chapters or parts, and 4, resp. 5 sections or groups. Out of these, belonging to Part I we achieved considerable progress in the group of questions 1.1 (see [1, 20, 23, 21, 22, 24, 25], in 1.2 [2], and in 1.4 [3, 4, 6, 8, 9], while the only group of questions where we have not been successful is 1.3 (minimal extrapolation problem). This group of questions was neglected by decision in view of the special circumstances arising lately (which we will describe below), so we concentrated on further development of the other topics (which is well-documented by the unplanned extra achievements), and consider this change of the project focus a necessary one, out of the range of our responsibility.
Regarding Part II, we achieved results in questions of in 2.2 [1], 2.3 [12, 13, 15, 14, 17, 7] and in 2.4 [7, 13, 16, 14, 17, 19], while not succeeding regarding 2.1 (Rendezvous numbers and the geometry of metric and Banach spaces) and 2.5 (Hypoelliptic partial differential operators). For not publishing about 2.1 and 2.5 the reason is simple: partially attracted by other, really intriguing other problems of the project, and partially due to our personal circumstances mentioned above, we did not have enough time to arrive at even these. This may be held against us, but we hope that the achievements in the other sections, many times surpassing the set goals, compensate also for the lack of results here. This is definitely a partial alteration from the research plan, which we admit and explain in the report, in particular as regards section 2.3, where losing the race for the first solution of the original conjecture of Ambrus–Ball–Erdélyi, aimed at originally, we were kind of driven into a much more extensive work to surpass the original aims (and thus to earn the right to publish in spite of the fall of the original conjecture).
Here we should also admit that our proven progress regarding section 2.3 (The real linear polarization constant problem) is below expectations and thus satisfying the set goals of the project proposal is not unquestionable here. However, regarding section 2.3 we at least did have some progress: below we have more to say about this. Also it is the very nature of our research work—particularly if we take into account the often emphasized "high risk-high gain" point of view of the research initiatives of such project supports—that with putting in the expectable amount of ideas, work and research efforts, in some questions very good results are obtained and in some problems less is found. In fact, the topic was a long-standing, well-studied, deep conjecture of functional analysis—and clearly if one does research in the direction of the Riemann Hypothesis, he could not be blamed if he does not succeed, and we still need to study, try, attack even the most intriguing problems. Therefore, although admitting our weaknesses in this regard, we consider this a case of a reasonable fluctuation of efficiency within the many questions addressed.

Looking for a numerical evaluation of the success rate, we may say that 6 out of 9 sections were dealt with achieving some successes, in most cases well over the set out aims, and in five out of the six at least delivering what was targeted, while in one section at least providing some progress towards the goals.

Let us note that not publishing or not delivering the set goals in some of the questions of the project plan does not mean that our work met unsurmountable obstacles, but in the most part it only signals the lack of enough time (available for research) under the circumstances and among the other multifaceted involvements we had encountered during these years. Nevertheless, we had made studies, contacted other researchers, were involved in various discussions among our team members and with other researchers, and what we have learnt strengthens our conviction that it is possible to address these goals by the methods we have in mind. (Of course, actual results can only be guaranteed when achieved—which we did not.)

As mentioned above, a special reason withhold us in addressing questions belonging to section 1.3. Namely, to this topic our attention was drawn by lectures of L. Zsidó (particularly his lecture when taking his seat as a member of our Academy), who in joint works with Haagerup, have addressed these kind of problems. However, as he explained to the project coordinator, they also had an ongoing joint work, almost finished, what they wanted to

close soon. At the time of the project writing it seemed a reasonable calculation that this publication will conclude in a short time, opening up the way for further research on the topic.
However, that publication has not appeared ever since. In the meantime, Haagerup unexpectedly died in an accident, and L. Zsidó was left alone to finish their project. Understandably, we did not feel like interfering with this delayed publication. We did have enough other questions to work on, whether planned or not, e.g., the new extremal problem of Soundararajan, or the Montgomery–Mockenhoupt Conjecture, so we instead were waiting for the completion of the said work by L. Zsidó. He was an invited speaker at our Fifth Workshop on Fourier Analysis in August 2015, but still he could only say about this that he plans to conclude and publish this treatise soon.

Let us describe how things stand regarding section 2.2 (Linear polarization constant problem), where our studies provided only some hopefully useful preliminary results. We pursued a two-sided approach, trying to combine methods and results developed for other purposes but possibly applicable also for the polarization problem. One side of this study is the general study of well-distributed point systems in any sense: e.g. in the sense of potential theory with some other kernels, or even in the sense of Euclidean distance. The first, potential theoretical issue is closely related to other constructions of special point systems with good distribution in some sense or in another, and also to the equilibrium problems, which we also have addressed. Most developed is, however, the study of Euclidean distance distribution properties on the unit ball of high dimensional spaces, since this is the heart of the matter of the design of self-correcting codes and densest sphere packing. Following the methods of Gorbachev, Cohn, Elkies and recently the Vanderbilt School of potential theory, currently the best results are due to refined one-sided approximation by weighted orthogonal polynomials of the kernel (which, in this case, is just the logarithmic kernel, i.e., the logarithm of the Euclidean distance). In particular, we found ways to invite Petar Boyvalenkov (twice) and Peter Dragnev from the latter school and got them explaining their methods and results in considerable detail: in particular, both of them delivered detailed lecture series at our Summer School in Potential Theory, while other top experts of ball packings, Dimitry Gorbachev and Nikolay Kuklin, were invited and listened to at the Fifth Workshop on Fourier Analysis. (Just one instance why it is incommensurably more advantageous to *organize* and *host* such meetings then just to participate: We could really invite and learn from the very experts we needed to listen to.)
The only "slight" change what we need is to replace this standard logarithmic kernel on the surface of the unit ball by the logarithm of the *inner product* of unit vectors. However small this change may seem, the difference is considerable: e.g., this new kernel becomes singular on an entire hyperplane, not only at just one point; it becomes insensitive to changes of x to $-x$ e.g., there is no *uniqueness* of the equilibrium measure at all etc. Still, we have in mind to adapt some of these one-sided approximation methods in the good hope of arriving at new results in the estimation of the polarization constant. Although the paper does not say a word about this long-term goal in mind, the recent article [1] presents a step forward in the preliminary study of the approach. That is the second leg of the slow, massive build-up, needed to carry out this technically more than involved approach: we started long-term cooperation with some Russian colleagues (and the research group of approximation theory in Yekaterinburg) with expertise in these type of one-sided approximation questions. Again, Marina Deikalova, Alexander Babenko and Vitalii Arestov were invited to our conferences, and Sz.R. visited them in Yekaterinburg twice (using the support of the OTKA grant), so that by the end of the year we could at least start a real collaboration on the mentioned relevant questions. That is why we think that the support of our project was not simply "lost" or "wasted" even regarding the problems set forth in the plan but not becoming ripe and suitable to harvest them to date.

Such examples can of course be continued. We have started collaborations e.g. on the Delsarte extremal problem, too, belonging to the group / section 1.1, sufficiently (in fact, quite over plans) researched anyway, but not leading to a publication in this particular problem, although it was also mentioned in the plan. Now we are close to finish a paper about that, too, but—partly due to the time restraints of our collaborator, who has moved from Oman to Germany recently and so could of course not continue her research work with the same energy as under settled, quiet circumstances—this still may need a few month more. Nevertheless, we are grateful for our OTKA grant to make it possible to invite and host Elena Berdysheva last year, with whom we could considerably progress on that matter. However, perhaps there is no need to account for these type of preliminary stage research work in so much detail: it should be clear that research is a continuous, long-term undertaking, progress in which is done over years in various directions until something becomes suitable for publication (and even then the matter is still in need of a continuous work of one or perhaps several years until the actual paper is published). We should just mention that we did study these questions, made various progresses and the support of our grant was instrumental also in that regard.

4. Some interesting findings and highlights of the project execution in detail

Extremal problems for positive definite functions. An achievement, which can be classified right into section 1.1 of the research plan, even if not formulated in the plan (as it was not expected at the time of writing), is a notable advance in an old and famous question of Hardy and Littlewood. The famous Hardy-Littlewood majorant problem was raised in the 30's and it can be formulated as the question whether $\int |f|^p \geq \int |g|^p$ whenever $\widehat{f} \geq |\widehat{g}|$. The answer is affirmative only for exponents p which are even integers. Montgomery conjectured that even among the idempotent polynomials there must exist some counterexamples, i.e., there exists some finite set of exponentials and some $\pm$ signs with which the signed exponential sum has larger p^{th} norm than the idempotent obtained with all the signs chosen $+$ in the exponential sum. That conjecture was proved in 2009 by Mockenhaupt and Schlag. However, a natural question is if even the classical three-term exponential sums $1 + e^{2\pi ix} \pm e^{2\pi i(k+2)x}$, used for $p = 3$ and $k = 1$ already by Hardy and Littlewood, should work in this respect for all p with $2k < p < 2k+2$.
Note that this was explicitly stated, but with a proof restricted only to the case of $k = 1$, and even then with some serious gap in the proof, already in the 1996 habilitation thesis of Mockenhoupt. Finally Mockenhaupt and Schlag arrived at a solution of the original conjecture of Montgomery in 2009 by employing a tricky construction of a four-term counterexample (coming from a product of two two-term exponentials, which product structure was essentially exploited), but this way the sharper question remained open.
S.K. attacked the problem in a series of papers [20, 23, 21], also included into his PhD Thesis [22] and *proved even the sharper conjecture of Mockenhaupt* for $k = 0, 1, 2, 3, 4, 5$; i.e., in the range $0 < p < 12$, $p \notin 2\mathbb{N}$. In case $k = 0$, i.e., $0 < p < 2$, he provides an analytical solution, based on the work of Bonami and Révész, while for the other cases various numerical methods, in particular numerical integral estimation procedures are used. Refinements in the technical features of the approach include use of total variation and integral mean estimates in error bounds for a certain fourth order quadrature. For $k = 5$, the estimates make good use of the special forms of the occurring functions. Thus the quadrature error estimates are less general, but one could get even better constants which are of practical use for the result.
About the ranking and the echo of the achievement we may note that the first paper [20] of the series *gained the Ames Award* of 2012, which prize is awarded by the American Mathematical Society and the Editorial Board of the Journal of Mathematical Analysis and Applications for the *best paper in the whole year*. In fact, there are two awards per year, one for theoretical results and one for applied mathematics: Krenedits obtained the prize (*worth of 2500 dollars*) for the best theoretical paper. This is a notable achievement, for the number of published papers is about one thousand, with *more than three thousand submissions per year*, and so the competition is rather wide.

In [24] S.K. and Sz.R. described the so-called general Carathéodory–Fejér extremal problems, constituting a wide-ranging extension of the century-old extremal problem on non-negative polynomials, studied by Carathéodory and Fejér. In this context, S.K. and Sz.R. considered the extremal problem of maximizing the function value $|f(z)|$ at a given point $z \in G$ attained by some positive definite and continuous function f on a locally compact group G, where for a given symmetric open set $\Omega \ni z$, f vanishes outside Ω and is normalized by $f(0) = 1$. They have extended the former results for classical groups to arbitrary locally compact Abelian (LCA) groups; thus unified the theory for many, until then isolated and repetitiously treated problems (cf. "pointwise" Turán problem, Boas-Kac problem, Carathédory–Fejér problem etc.). But the really interesting advance occurred only when they tried to extend their methods further, to not necessarily commutative locally compact groups. In fact, the structure of l.c. groups in general is still not fully understood, and there are many notions, constructions, classes etc. dealt with. On our part, we have found a condition, what the extension really required, and thus defined *roundness* of an l.c. group element $z \in G$. It took several years until it turned out that this notion is actually *equivalent* to another notion, studied in the theory of l.c. groups, namely to the property of *equicontinuity* (which is not a too good name, though) of the point $z \in G$. Thus finally we can formulate the extended result as something described by well-established terms of l.c. groups theory, while providing a proper extension (covering all Abelian cases) to the l.c.a. settings.

Investigation of asymptotic behavior and order of oscillation of functions by Fourier analysis. To explain our achievements regarding section 1.2, we should remind to the context of the classical Wiener–Ikehara Tauberian theorem, in which the condition of monotonicity were extended with the generalized condition of slow decrease and perhaps some additional poles on the boundary of convergence of the Laplace transform. In this generality, we have proved the otherwise known asymptotic evaluation of the transformed function, when the usual conditions of the Wiener–Ikehara theorem hold. However, our version also provides an effective error term, which was not derived thus far for this general case. The crux of the proof is a proper variation of the lemmas of Ganelius and Tenenbaum, also constructed for the sake of an effective version of the Wiener–Ikehara Theorem.

The point here is that one should handle simultaneously a weak, yet rather uncontrollable oscillation of the function, together with an effective error bound in the (already complicatedly depending) Fourier or Laplace transform. A way to squeeze out an effective error estimate was devised by Tenenbaum, by modifying appropriately the Lemma of Ganelius—but on the other hand, he had to assume that the function is monotonic. Clearly to handle the general case an effective control of even the oscillating term was needed, and even Tenenbaum himself viewed this impossible unless using rather heavy restrictions like e.g. a certain analytic expansion etc. Therefore it came as a surprise when a really effective result was indeed found. Moreover, the analysis also revealed that together with the well-established classes of slow increasing functions, there are other, non-equivalent definitions and notions of moderate or well-controlled decrease, which may as well be subject of similar effective results.

Periodic decomposition of functions. Classically, periodic functions can be studied via their Fourier series. The original problem we started with almost three decades ago, was the seemingly innocent question, due to Ruzsa, whether a bounded and continuous function on $\mathbb{R}$, $f \in C_b(\mathbb{R})$ can be written as the sum of n continuous, periodic functions (periodic by given $a_1, \dots, a_n \in \mathbb{R}$), if the difference equation $(S_{a_1} - I)\dots(S_{a_n} - I)f = 0$ holds. Here $S_a f(x) := f(x+a)$ is the shift operator (by a). This question was answered by Révész and Laczkovich in the affirmative. This innocently looking question initiated an avalanche of research papers, and its ramifications have become countless. By the nature of the problem an immediate generalization in operator theory is evident: ask the same equality between the kernel of the product and the sum of the kernels when some general linear operators on some Banach space replace the shift-operators:

$$\text{Does the equality} \quad \ker(T_1 - I)\dots(T_n - I) = \ker(T_1 - I) + \dots + \ker(T_n - I) \quad \text{hold true?}$$

We have made the following progress in the proposed research questions. We could settle the case when $T_j = T(t_j)$, for some strongly continuous Banach-space representation T of $\mathbb{R}$ (provided E does not contain an isomorphic copy of the Banach space c_0 of null-sequences), or when T is a strongly continuos semigroup-representation of $[0, \infty)$ that is asymptotically norm continuous, see [4, 9]

On this way we have discovered the following beautiful fact: A separable Banach space E does not contain a copy of the space c_0 of null-sequences if and only if for every doubly power-bounded operator T on E and for every vector $x \in E$ the relative compactness of the sets $\{T^{n+m}x - T^n x : n \in \mathbb{N}\}$ (for some/all $m \in \mathbb{N}$, $m \geq 1$) and $\{T^n x : n \in \mathbb{N}\}$ are equivalent. With the help of the Jacobs–de Leeuw–Glicksberg decomposition of strongly compact semigroups similar results can be obtained for non-invertible, power-bounded operators, see [6].

The original problem (and our versions) can be generalized to commuting actions of groups and semigroups, replacing the translation action of $\mathbb{R}$ (or $[0, \infty)$. We have achieved positive results when the actions are amenable, or when the acting semigroup of bounded linear operators is mean-ergodic, see [4].

Our work culminated in [9] where we survey about the known results and reveal their connections to harmonic analysis, operator theory, functional analysis, Banach space geometry, ergodic theory, and give new and short proofs for classical result, such as the above mentioned one of Laczkovich and Révész, showing the power of the operator-ergodic theoretic approach.

Much of the background material that was needed for this project is contained in [3].

Chebyshev constants, energies, transfinite diameter and applications. One of the most successfully explored group of research questions (where we had gone far beyond the set goals) were those of section 2.3 of the plan. To explain some of these results, let us start with the paper [15]. On the fundament of the classical theory there was based the abstract *linear* potential theory, where $\mathbb{R}^3$ is replaced by some locally compact space X, and the Newtonian kernel by some lower semicontinuos kernel function, $k(x,y) : X \times X \to \mathbb{R} \cup \{\infty\}$. This theory is developed by G. Choquet, B. Fuglede, M. Ohtsuka, M. Yamasaki, L. Carleson and N.S. Landkof, etc. However, *two types of normalizations* can be found in the literature: normalization with respect to μ (i.e., $\mu(K) = 1$) and with respect to the constant level value W of the equilibrium potential on the set K (i.e., $W = 1$). The second one leads to *non-linear* potential theory. While the linear capacity of a set is $C(E) = W(E)^{-1}$, the p-capacity can be defined as $C_p(E) := \inf\{\int_X f^p d\nu : f \geq 0, \quad f \in L_{\nu,p}, \quad \int_X k(x,y)f(y)d\nu(y) \geq 1 \quad \forall x \in E\}$, where ν is a fixed measure on X.

The theory of classical potentials the theory is connected with potentials of polynomials and hence approximation theory and interpolation theory, and this relationship has a wide practical importance. Namely, making appropriate conditions on the kernel, the transfinite diameter, the Chebyshev constant and the capacity of a compact set coincide. Actually, these notions are the limits of the finite-set versions of potential and energy. After the classical investigations of, e.g., M. Fekete and J. Siciak, at the end of the 90s several applications were inspired by the monograph of E. Saff and V. Totik on logarithmic potentials with external fields. Transfinite diameter and Chebyshev

constant in locally compact spaces were examined by B. Farkas and B. Nagy. These investigations gave chance of defining some greedy energy points which are asymptotically as good as the minimal energy (or Fekete) points, but less difficult to compute them. The power of this discretization method is well illustrated by the applications in metric spaces, for instance computing Hausdorff measures. It was also pointed out that Fekete sets are optimal from the point of view of interpolation. Now the paper [15] introduces analogous notions, extending the notions of transfinite diameter and Chebyshev constant to the non-linear case. On behalf of dispersion of minimal energy points we will take into consideration the convex combination of linear and non-linear energy. If the kernel is infinite at the diagonal, the common infimum will ensure the dispersion of the points. The transfinite diameter and (Wiener-type) energy are defined and besides the equivalence, certain properties of the energy are studied.
These considerations yield non-symmetric kernels which generate some difficulties in connection with the definition of the potential. First, it is given and examined on "one level", with respect to a measurable function f, and only then independently of any functions. The properties of the equilibrium potential are also given. The definition of potential gives the possibility of defining the Chebysev constant and greedy energy points. Finally, some observations are made with respect to symmetry and the behavior of the notions are examined when the convex combination approaches one of the endpoints.

A good example of a successful investigation of the original research goals may be the paper [12]. To explain it in more detail let us recall that L. Fejér introduced the so-called Hermite–Fejér interpolatory process, and in 1934 he gave the definition of normal- and ρ-normal system of nodes for which the Hermite–Fejér interpolation is a positive interpolatory process. The surprisingly nice convergence properties of Lagrange, Hermite and Hermite–Fejér operators on ρ-normal systems were proved by L. Fejér, G. Grünwald, etc. On the other hand there are other possibly good systems of nodes: the set of Fekete points (i.e., n^{th} diameter extremal points), which has uniform distribution in the proper sense (leading to the transfinite diameter in the limit), hence it must be a good node system for interpolation. Note that some properties of the Fekete points are very similar to the properties of the system of zeros of orthogonal polynomials, extensively used to construct interpolatory processes. From another point of view, Egerváry and Turán asked, if it is possible to find an interpolatory process, and a system of nodes together, where the interpolatory polynomial has the minimal degree, and the operator has the minimal norm. The above-mentioned point systems could possibly be suitable systems of nodes for an interpolatory process in the general sense and also with respect to the Egerváry–Turán problem.
The paper [12] reconsiders the connections among these systems of nodes and the respective interpolatory problems, summarize, reformulate and complete these results. It is also pointed out, that in these equivalences the so-called Fejér constants, introduced in this very paper, play a key role, since the characterization of these special systems is ensured by the Fejér constants. So far the paper may be considered as a study of the relevant extremal problems and point sets, belonging to subsection 2.3 of the research plan.
But further in the third section, applying the above, a convergence theorem on a new weighted Grünwald type interpolatory process (on the real line and for Freud-type weights) is proved. As it turns out, giving the weighted Fekete sets with respect to a fixed weight is difficult. (However, there are several methods of giving approximating Fekete sets.) The zeros of orthogonal polynomials are Fekete sets for some varying weights. Unfortunately these varying weights tend to zero locally uniformly, so interpolation on Fekete sets in this sense gives only trivial (convergent) processes. The investigation of these weights at infinity leads to define a weighted Grünwald operator, which has rather good convergence properties. Comparing this result with the result of Lubinsky and Szabó it turns out that the convergence is valid here for a wider function class. So, these results provide a good example of how the somewhat abstract extremal problems, extracted from various concrete questions, have in turn applications in weighted approximation theory, targeted in section 2.3.

Applications of general potential theory in weighted approximation theory. One of the high points of our joint research work was the completion of the research work [7]. In this regard the original aim, set forth in the project plan, was to bring down a conjecture of Ambrus, Ball (and Erdélyi, who later joined to the publication): sums of translates of a concave kernel on the torus $\mathbb{T}$ (with one singularity at 0 and satisfying some reasonable assumptions) achieves global minimization of its maximum exactly for regular n-gons on the circle. After working for years, the group of researchers of the strong potential theory school at Vanderbilt University was cutting ahead of us publishing the solution of the original conjecture.
Still, we worked further and obtained a much more general, also rather more extensive and throughout description of the whole problem, including e.g. *a full solution of the analogous question when the translated kernels can all be different*. This paper was finally uploaded to the arxiv as a preprint right on December 31, 2015 (although with most of the results being clarified somewhat before, we had already delivered lectures about our work).

Note that these results not only go well over the original aims of section 2.3 (and section 2.4) of the research plan—what was "only" to prove the original conjecture of Ambrus, Ball and Erdélyi, finally proven ahead of us by Hardin, Kendall and Saff in 2013—but apart from the considerable extension in the original theme, we also solve (as an illustrative example of applicability) a generalization and extension of a famous approximation theory problem of B. Bojanov. Bojanov studied the question of minimal maximum norm of a monic degree n polynomial P on the interval $[-1,1]$, provided all the zeroes are having prescribed multiplicities: i.e., there are given integers $k \in \mathbb{N}$ and $\nu_j \in \mathbb{N}$ $(j = 1,\dots,k)$, such that $\nu_1 + \cdots + \nu_k = n$, and $P(x) = \prod_{j=1}^{k}(x - x_j)^{\nu_j}$. Bojanov's method in proving existence and uniqueness, as well as equioscillation of the extremal polynomial, followed the classical, Markov-type variational, zero-counting processes, which is quite complicated and tedious. In particular, this was never worked out for the analogous, but technically even more complicated case of trigonometric polynomials, and was neither applicable for the study of the so-called "generalized polynomials" (in the sense of Borwein and Erdélyi). Now with our equilibrium-type methods and results we could find a direct, shorter way to settle this problem, including trigonometric and generalized polynomials, too. This application, also written up in the same paper [7], should belong already to section 2.4 of the research plan, but providing an absolutely unplanned, unforeseen result enriching the content of this group of research questions.

5. Scientific meetings — a way of leverage in the utilization of the support

Naturally, achievements of a research team can be measured only partly by publications. It belongs to the successes of the project that with the support of the OTKA grant we were able to participate on conferences and present our results to a wide public of researchers. Let us particularly mention here the Fourier Analysis Workshops (in 2013 and also in 2015) and the 10th Summer School in Potential Theory (2015), all organized (not exclusively, but in a great deal) by our team members in Hungary. These occasions served the wider acceptance and prestige of the Hungarian school of mathematics well, and brought useful connections and exchanges not only to our own group, but to several colleagues. Organization by and massive participation of our team members were instrumental for the success of these conferences, so however indirectly but we can say that the OTKA support made it possible to organize and successfully conduct these international meetings. On the other hand, as explained above, the advantages of our OTKA supported participation could be coupled by the influence we could exercise in the invitation of speakers and the setting up the program to benefit the most regarding our advance in the directions of our project goals.

6. Summary

To give an overall summary of the project, we could say that the support of OTKA directly helped us to publish new scientific research material of well over a thousand pages, as well as to continue our quest for solutions of research questions not arriving to a conclusion during the given time frame of the project funding. The publication of a book in Springer's Graduate Texts in Mathematics series, a PhD dissertation, gaining an Ames Award, publishing altogether 27 items, mostly in good journals, and massively participating in the conduction of three high-level international conferences in Budapest, were altogether made possible through this support.

According to our own evaluation, we succeeded considerably in about 70% of the questions, posed in our preliminary research plan five years ago, in some cases going well over the set goals: also a good deal of related, newly arising problems of similar nature than in the project were solved, thus modifying the project content in about an extent of 30%. For what particular questions a count of closed cases, i.e., publications are missing from our final report, is roughly compensated by the extra achievements, also achieved by use of the OTKA grant support. We on our part consider that our research project was worthy of the support, and concluded successfully—and do hope that the jury of the newly organized NKFIH will share our judgement.