Following our research plan, we have mainly done research – and established a number of significant results – in several areas of set theory:

I. Combinatorics

II. Cardinal invariants of the continuum and ideal theory

III. Set-theoretic topology

In addition to these, G. Sági has done extended research in model theory that had ramifications to combinatorics.

We presented our results in 38 publications, almost all of which appeared or will appear in the leading international journals of these fields (5 of these papers have been submitted but not accepted as yet). We also participated at a number of international conferences, three of us (Juhász, Sági, Soukup) as plenary and/or invited speakers at many of these. We now give an overview of our results.

I. INFINITARY COMBINATORICS

The "Handbook of set theory" was being prepared by the leading experts of set theory for more than ten years. The very long chapter [14] contains the most significant modern results on partition relations, a branch of set theory that was created by Paul Erdös and Richard Rado in the early fifties. The first fifty pages of [14] – that was actually written by me – cover the results for infinite resources. The proofs make very extensive use of elementary submodels and yield somewhat stronger results than was known before. Here is a list of some of the most interesting of them:

– Generalisations of the 1991 Baumgartner, Hajnal, Todorcevic theorem (in Section 4),

– Shelah's remarkable positive result for the case of infinitely many colors from 2003 (in Section 6),

– the theorems for polarized partitions in cases of successors of weakly compact and measurable cardinals in Section 7 and Section 8, respectively.

Erdős and Rado proved that there are triangle-free graphs with arbitrarily large chromatic number. On the other hand, later Erdős and Hajnal showed that every uncountably chromatic graph contains copies of all finite bipartite graphs, but They also proved that for each natural number n there is an uncountably chromatic graph that omits all cycles of odd length up to $2n+1$. This determines which finite graphs must occur in every uncountably chromatic graphs: these are precisely the bipartite graphs.

Erdős and Hajnal then started to generalize these results by investigating the corresponding problem for r-hypergraphs, for $3 \leq r < \omega$. A long article of P. Erdős, F. Galvin and A. Hajnal was devoted to this question. We revisited some problems which were left open in that paper. In a joint paper with Péter Komjáth, [2], a class of triple systems is determined such that each of these systems must occur in every triple system with uncountable chromatic number that omits $\mathcal{T}_0$, the unique triple system with two triples on four vertices. This class contains all circuits of odd length ≥ 7, where a circuit of length n is a triple system of the form $\{x_0, x_1, y_0\}, \{x_1, x_2, y_1\}, \ldots, \{x_{n-1}, x_0, y_{n-1}\}$ with the different variables pairwise distinct. It is also shown that it is consistent that there are two finite triple systems that can be separately omitted by uncountably chromatic triple systems, but not both.

In [3] we intended to investigate the relationship between some theorems in finite combinatorics and their infinite counterparts: given a "finite" theorem how one cat get a "infinite" version of it? So we studied the methods of generalizations. We will

survey some problems from finite combinatorics and we will analyze the relationship between their proofs and the proofs of their "infinite" versions.

Besides these, the paper gave a proof of a theorem of Erdős, Grünwald and Vázsonyi giving the full descriptions of graphs having one/two-way infinite Euler lines. The last section contained some new results: an infinite version of a multiway-cut theorem was included.

The aim of [31] was to explain how to use elementary submodels to prove new theorems or to simplify old proofs in infinite combinatorics. The paper mainly addresses novices learning this technique: we introduced all the necessary concepts and gave easy examples to illustrate our method, but the paper also contained new proofs of theorems of Nash-Williams on decomposition of infinite graphs, and an improvement of a decomposition theorem of Laviolette concerning bond-faithful decompositions.

In the first 3 sections we recalled and summarized all necessary preliminaries from set theory, combinatorics and logic, then we gave the first application of elementary submodels, and we explained why it is natural to consider Σ-elementary submodels for some large enough finite family Σ of formulas. Next we used elementary submodels to prove some classical theorems in combinatorial set theory. All these theorems have the following Ramsey-like flavor: *Every large enough structure contains large enough "nice" substructures.*

In the second part of the paper we proved structure theorems of a different kind: *Every large structure having certain properties can be partitioned into small "nice" pieces.* A typical example is Nash-Williams's theorem on cycle decomposition of graphs without odd cuts. To prove these structure theorems it is not enough to consider just one elementary submodel but we should introduce the concept of *chains of elementary submodels.*

Finally, we gave a more elaborate application of chains of elementary submodels to eliminate GCH from a theorem concerning bond-faithful decomposition of graphs.

Given an almost disjoint family $\mathcal{A} \subset [\omega]^\omega$, one can guess that we can always find an almost disjoint family $\mathcal{B} \subset [\omega]^\omega$ of size at most $\mathfrak{a}$ such that $\mathcal{A} \cup \mathcal{B}$ is a maximal almost disjoint family. However, it is not the case, as the following surprising results of Kunen shows: there is an almost disjoint family $\mathcal{A} \subset [\omega]^\omega$ of size 2^ω such that if $\mathcal{A} \cup \mathcal{B} \subset [\omega]^\omega$ is a maximal almost disjoint family, then $|\mathcal{B}| = 2^\omega$.

In [25] we defined $\mathfrak{a}^+(\kappa)$ as to be the minimal cardinal μ such that if $\mathcal{A} \subset [\omega]^\omega$ is an almost disjoint family of size κ, then there is an almost disjoint family $\mathcal{B} \subset [\omega]^\omega$ of size at most μ such that $\mathcal{A} \cup \mathcal{B}$ is a maximal almost disjoint family. Using this notation, Kunen's result is the equality $\mathfrak{a}^+(2^\omega) = 2^\omega$.

In [25] we showed that the inequalities $\aleph_1 = \mathfrak{a} < \mathfrak{a}^+(\aleph_1) = 2^\omega$ and $\mathfrak{a} = \mathfrak{a}^+(\aleph_1) < 2^\omega$ are both consistent. Especially, the first one holds in the Cohen model. We also gave several constructions of mad families with some additional properties.

Let $\mathbb{D}$ denote the partially ordered sets of homomorphism classes of finite directed graphs, ordered by the homomorphism relation. An *(in)finite-(in)finite duality pair* in $\mathbb{D}$ is a partition (B, C) of an antichain $B \cup C$ such that B is (in)finite, C is (in)finite, and the up-set of B and the down-set of C cover $\mathbb{D}$.

Nesetril and others gave full descriptions of finite-finite duality pairs. An earlier paper we made the following easy observation: there are continuum many infinite-infinite duality pairs.

It is still open whether there are infinite-finite duality pairs. However, in [11] we could prove that finite-infinite duality pairs do not exist.

– conflict free colorings

A coloring of a set-system $\mathcal{A}$ (formally, a map f defined on $\cup\mathcal{A}$) is called *conflict free* if every member of $A \in \mathcal{A}$ has a point whose color differs from the color of any other point in A. The *conflict free chromatic number* $\chi_{\mathrm{CF}}(\mathcal{A})$ of $\mathcal{A}$ is the smallest ρ for which $\mathcal{A}$ admits a conflict free coloring with ρ colors. Clearly, if all elements of $\mathcal{A}$ have size > 1 (that we always assume) then no member of A is monochromatic for a conflict free coloring, hence the chromatic number $\chi(\mathcal{A}) \le \chi_{\mathrm{CF}}(\mathcal{A})$.

$\mathcal{A}$ is a (λ, κ, μ)-system if $|\mathcal{A}| = \lambda$, $|A| = \kappa$ for all $A \in \mathcal{A}$, and $\mathcal{A}$ is μ-almost disjoint, i.e. $|A \cap A'| < \mu$ for distinct $A, A' \in \mathcal{A}$. Erdős and Hajnal investigated the chromatic numbers of (λ, κ, μ)-systems in the 60's and our aim in [26] was to run a parallel study of

$$\chi_{\mathrm{CF}}(\lambda, \kappa, \mu) = \sup\{\chi_{\mathrm{CF}}(\mathcal{A}) : \mathcal{A} \text{ is a } (\lambda, \kappa, \mu)\text{-system}\}$$

for $\lambda \ge \kappa \ge \mu$, actually restricting ourselves to $\lambda \ge \omega$ and $\mu \le \omega$. It turned out that the three cases 1.) $\omega > \kappa \ge \mu$, 2.) $\kappa \ge \omega > \mu$, and 3.) $\omega = \mu$ require very different methods. Here is a list of our main results:

(1) for any limit cardinal κ (or $\kappa = \omega$) and integers
$n \ge 0$, $k > 0$, GCH implies

$$\chi_{\mathrm{CF}}(\kappa^{+n}, t, k+1) = \begin{cases} \kappa^{+(n+1-i)} & \text{if } i \cdot k < t \le (i+1) \cdot k, \ i = 1, ..., n; \\ \kappa & \text{if } (n+1) \cdot k < t ; \end{cases}$$

(2) if $\lambda \ge \kappa \ge \omega > d > 1$, then $\lambda < \kappa^{+\omega}$ implies $\chi_{\mathrm{CF}}(\lambda, \kappa, d) < \omega$ and $\lambda \ge \beth_\omega(\kappa)$ implies $\chi_{\mathrm{CF}}(\lambda, \kappa, d) = \omega$;

(3) GCH implies $\chi_{\mathrm{CF}}(\lambda, \kappa, \omega) \le \omega_2$ for $\lambda \ge \kappa \ge \omega_2$ and
V=L implies $\chi_{\mathrm{CF}}(\lambda, \kappa, \omega) \le \omega_1$ for $\lambda \ge \kappa \ge \omega_1$;

(4) the existence of a supercompact cardinal implies
the consistency of GCH plus
$\chi_{\mathrm{CF}}(\aleph_{\omega+1}, \omega_1, \omega) = \aleph_{\omega+1}$ and
$\chi_{\mathrm{CF}}(\aleph_{\omega+1}, \omega_n, \omega) = \omega_2$ for $2 \le n \le \omega$;

(5) CH implies $\chi_{\mathrm{CF}}(\omega_1, \omega, \omega) = \chi_{\mathrm{CF}}(\omega_1, \omega_1, \omega) = \omega_1$, while
MA_{ω_1} implies $\chi_{\mathrm{CF}}(\omega_1, \omega, \omega) = \chi_{\mathrm{CF}}(\omega_1, \omega_1, \omega) = \omega$.

– Rainbow Ramsey Theory

Anti Ramsey (polychromatic Ramsey, rainbow Ramsey) theory deals with the following kind of problems: given a coloring f of certain subsets of a set X can you find a large subset Y of X such that f is inhomogeneous (e.g. injective) on the colored subsets of Y? Obviously, to get positive results we should have some assumption concerning the coloring f.

Erdős's first rainbow Ramsey question (Problem 68) was the following problem: *Assume that c establishes that $\omega_1 \nrightarrow [\omega_1]_\tau^2$ with more than 2 colors. Does there exist a rainbow triangle?*

It was known that if c establishes that $\omega_1 \nrightarrow [\omega, \omega_1]_3^2$, then there exists a rainbow triangle. Shelah proved that it is consistent that some c establishes that $\omega_1 \nrightarrow [\omega_1]_\tau^2$ without a rainbow triangle. On the other hand, Erdős and Hajnal proved , that if c establishes just $\omega_1 \nrightarrow [\omega_1; \omega_1]_3^2$, then f realizes each function $d : [\omega]^2 \to \omega$, especially there is an infinite rainbow subset containing all the colors.

Recently, in [1], we revisited these problems because in the last decades Todorcevic and Moore developed new methods to construct colorings with some strong properties. We proved that if c establishes just $\omega_1 \nrightarrow [\omega_1, \omega_1]_3^2$, then there is an infinite rainbow set, but it is not necessarily true that f realizes each function $d : [\omega]^2 \to \omega$.

These theorems considered colorings of the pairs of ω_1. Can we "step up" to get similar theorems for ω_2? Especially, Erdős and Hajnal asked the following: *Assume that c establishes $\omega_2 \nrightarrow [\omega_1, \omega_2]_2^2$. Does f realize each function $d : [\omega_1]^2 \to 2$?*

In the first part of the [10] we showed that if a coloring c establishes $\omega_2 \nrightarrow [(\omega_1 : \omega)]_2$ then c establishes this negative partition relation in each Cohen-generic extension of the ground model, i.e. this property of c is Cohen-indestructible. This result yields a negative answer to a the above mentioned question of Erdős and Hajnal: it is consistent that GCH holds and there is a coloring $c : [\omega_2]^2 \to 2$ establishing $\omega_2 \nrightarrow [(\omega_1 : \omega)]_2$ such that some coloring $g : [\omega_1]^2 \to 2$ is not realized by c .

It is also consistent that 2^{ω_1} is arbitrarily large, and there is a function g establishing $2^{\omega_1} \nrightarrow [(\omega_1, \omega_2)]_{\omega_1}$; but there is no uncountable g-rainbow subset of 2^{ω_1}.

In the second part of [10] we dealt with rainbow Ramsey theorems in which we had a different type of restriction concerning our colorings. Instead of establishing negative partition relations we assumed that our colorings are "bounded": a function $f : [X]^n \to C$ is μ-*bounded* iff $|f^{-1}\{c\}| \le \mu$ for each $c \in C$.

We showed that if GCH holds then for each $k \in \omega$ there is a k-bounded coloring $f : [\omega_1]^2 \to \omega_1$ and there are two c.c.c posets $\mathcal{P}$ and $\mathcal{Q}$ such that

$$V^{\mathcal{P}} \models \text{``} f \text{ c.c.c-indestructibly establishes } \omega_1 \nrightarrow^* [(\omega_1; \omega_1)]_{k-bdd}\text{''},$$

but

$$V^{\mathcal{Q}} \models \text{`` } \omega_1 \text{ is the union of countably many } f\text{-rainbow sets ''}.$$

– splitting covers

Let X be a set, κ be a cardinal number and let $\mathcal{H}$ be a family of subsets of X which covers each $x \in X$ at least κ times. What assumptions can ensure that $\mathcal{H}$ can be decomposed into κ many disjoint subcovers?

In [21] we examined this problem under various assumptions on the set X and on the cover $\mathcal{H}$: among other situations, we consider covers of topological spaces by closed sets, interval covers of linearl y ordered sets and covers of $\mathbb{R}^n$ by polyhedra and by arbitrary convex sets. We focus on these problems mainly for infinite κ. Besides numerous positive and negative results, many questions turn out to be independent of the usual axioms of set theory.

Our investigations were initiated by the question of J. Pach whether any infinite-fold cover of the plane by axis-parallel rectangles can be decomposed into two disjoint subcovers. After answering this question in the negative for ω-fold covers, we started a systematic study of splitting infinite-fold covers in the spirit of J. Pach et al.; in the present paper we would like to publish our first results and state numerous open problems.

We have organized the paper to add structure as we go along. First, for any pair of cardinals κ and λ, we studied the splitting of covers of κ by sets in $[\kappa]^{\le \lambda}$. Then we discuss the splitting of edge-covers of finite or infinite graphs. In the remaining sections of the paper we studied covers by convex sets. We showed that a cover of a linearly ordered set by convex sets is "maximally" decomposable. After completing our work, it turned out that R. Aharoni, A. Hajnal and E. C. Milner obtained results earlier which are similar to our results. Since our proofs are significantly simpler and yield slightly stronger results we decided not to leave them out.

As a preliminary study to covers by convex sets on the plane, we showed that the splitting problem for covers by closed sets is independent of ZFC. Roughly speaking, under Martin's Axiom an indecomposable cover of $\mathbb{R}$ can be obtained even by the translates of one compact set; while in a Cohen extension of a model with GCH, every uncountable-fold cover by closed sets is "maximally" decomposable. From these results we easily got that the splitting problem for covers of $\mathbb{R}^n$ by convex sets is independent of ZFC. This independence was accompanied by two ZFC results. We show that for very general classes of sets, including e.g. polyhedra, balls or arbitrary affine varieties, an uncountable-fold cover by such sets is "maximally" decomposable. On the other hand, we construct an ω-fold cover of the plane by closed axis-parallel rectangles which cannot be decomposed into two disjoint subcovers. We closed the paper with a collection of open problems.

II. Cardinal invariants of the continuum and ideal theory

The *cofinality spectrum* of the group of all permutation of natural numbers, $\mathrm{CF}(Sym(\omega))$, is the set of regular cardinals λ such that $Sym(\omega)$ is the union of an increasing chain of λ proper subgroups. Shelah and Thomas showed that $CF(Sym(\omega))$ cannot be an arbitrarily prescribed set of regular uncountable cardinals: if $A = \langle \lambda_n : n \in \omega \rangle$ is a strictly increasing sequence of elements of $\mathrm{CF}(Sym(\omega))$, then $\mathrm{pcf}(A) \subseteq \mathrm{CF}(Sym(\omega))$. On the other hand, they also showed that if K is a set of regular cardinals which satisfies certain natural requirements, then $CF(Sym(\omega)) = K$ in a certain c.c.c generic extension.

In [30] we investigated the *additivity spectrum* of certain ideals in a similar style. The additivity spectrum $\mathrm{ADD}(\mathcal{I})$ of an ideal $\mathcal{I} \subset \mathcal{P}(I)$ is the set of all regular cardinals κ such that there is an increasing chain $\{A_\alpha : \alpha < \kappa\} \subset \mathcal{I}$ with $\cup_{\alpha < \kappa} A_\alpha \notin \mathcal{I}$.

Assume that $\mathcal{I} = \mathcal{B}$ or $\mathcal{I} = \mathcal{N}$ or $\mathcal{I} = \mathcal{M}$, where $\mathcal{B}$ denotes the σ-ideal generated by the compact subsets of the Baire space ω^ω.

We showed that if A is a non-empty progressive set of uncountable regular cardinals and $\mathrm{pcf}(A) = A$, then $\mathrm{ADD}(\mathcal{I}) = A$ in some c.c.c generic extension of the ground model. On the other hand, we also proved that if A is a countable subset of $\mathrm{ADD}(\mathcal{I})$, then $\mathrm{pcf}(A) \subset \mathrm{ADD}(\mathcal{I})$.

For countable sets these results give a full characterization of the additivity spectrum of $\mathcal{I}$: a non-empty countable set A of uncountable regular cardinals can be $\mathrm{ADD}(\mathcal{I})$ in some c.c.c generic extension iff $A = \mathrm{pcf}(A)$.

The study of analytic P-ideals is a central topics in the recent set-theoretic investigation of the reals.

In [4] we studied some cardinal invariants of analytic P-ideals and some forcing properties of these ideal.

We gave upper and lower bound so the almost disjointness numbers of certain analytic P-ideals. Extending a result of Kunen, we showed that for any analytic P-ideal I, there exists an uncountable Cohen-indestructible I-mad family. Using Galois-Tukey equivalence of certain relation, we proved that the bounding number $\mathfrak{b}(I)$ and the dominating number $\mathfrak{d}(I)$ are independent from the ideals, they are always $\mathfrak{b}$ and $\mathfrak{d}$, respectively.

In the last part of this paper we investigated forcing properties of analytic P-ideals. Especially, if $\mathcal{Z}$ denotes the the density zero ideal, then (i) a poset $\mathbb{P}$ is $\mathcal{Z}$-bounding iff it has the Sacks property, and (ii) if $\mathbb{P}$ adds a slalom capturing all ground model reals then $\mathbb{P}$ is $\mathcal{Z}$-dominating.

The celebrated theorem of Hechler claims that if Q is a σ-directed poset in V, then in some ccc generic extension of the ground model some cofinal subset of

$\langle \omega^\omega, \leq^* \rangle$ is order isomorphic to Q. This result serves as a basic tool to investigate the behavior of certain cardinal invariants of the reals.

In [24] we gave a far-fetched genaralization of this theorem: if Q is a σ-directed poset in V, then in some ccc generic extension of the ground model every tall analytic P-ideal which has a code in the ground model has a cofinal subset which is order isomorphic to Q.

In [12] we proved that if P is a forcing notion in the ground model and $\mathcal{A}$ is an infinite almost disjoint family, then $\mathcal{A}$ can be extended to a P-indestructible MAD family in some ccc forcing extension of the ground model. Recently, in his PhD dissertation Farkas generalized this result for idealized almost disjoint families and for the random forcing.

In [32] we introduced a method for associating cardinal invariants to ideals by using (classical) partial orders on the set of all ideals on natural numbers. The Katetov-invariant of the density zero ideal was motivated by analytic consideration, namely by sequential properties of spaces of probability measures with the weak* topology.

We had results both on combinatorial properties of these cardinals and on their possible values in forcing extensions. Furthermore, we investigated the associated maximality properties of almost disjoint families and towers, and we proved some consistency results by using the Martin's Axiom for sigma-centered posets.

III. Set-theoretic topology

– compact spaces

The study of the important class of **compact spaces** traditionally has occupied a central place in our investigations. Some of our earlier investigations led to the quite extensive general study of the convergence and character spectra of compacta in [15] The convergence spectrum $cS(X)$ of a space X is the set of all sizes of converging (one-to-one) sequences in X, while the character spectrum $\chi S(X)$ is the set of all characters of (non- isolated) points in **subspaces** of X. For compacta (that we are really interested in) we always have $cS(X) \subset \chi S(X)$. Here is a selection of the results of [15] (X is always a compactum):

(1) If $\chi(X) > 2^\omega$ then $\omega_1 \in \chi S(X)$ or $\{2^\omega, (2^\omega)^+\} \subset \chi S(X)$.

(2) If $\chi(X) > \omega$ then $\chi S(X) \cap [\omega_1, 2^\omega] \neq \emptyset$.

(3) If $\chi(X) > 2^\kappa$ then $\kappa^+ \in cS(X)$, in fact there is a converging discrete set of size κ^+ in X.

(4) If we add λ Cohen reals to a model of GCH then in the extension for every $\kappa \leq \lambda$ there is X with $\chi S(X) = \{\omega, \kappa\}$. In particular, it is consistent to have X with $\chi S(X) = \{\omega, \aleph_\omega\}$.

(5) If all members of $\chi S(X)$ are limit cardinals then

$$|X| \leq (\sup\{|\overline{S}| : S \in [X]^\omega\})^\omega.$$

(6) It is consistent that 2^ω is as big as you wish and there are arbitrarily large X with $\chi S(X) \cap (\omega, 2^\omega) = \emptyset$.

The last item (6) shows that the character spectrum of a non-first countable compactum may (consistently) omit ω_1, the first uncountable cardinal, but it was left open in [15] if the convergence spectrum can do that. Item (3) implies that this may only happen if $\chi(X) \leq \mathfrak{c} = 2^\omega$. This problem turned out to be very hard and it took us in [6] a lot of work to construct, with a very complicated forcing argument,

a compactum X such that $cS(X) = \{\omega, \omega_2\}$. So far, this is the only known (consistent) example of a non-first countable compactum whose convergence spectrum omits ω_1.

A celebrated reflection theorem of A. Dow states that if every subspace of cardinality ω_1 of a compact space X is metrizable then so is X. Arhangelskii asked if this is also true for locally compact spaces and in [13] we proved that the answer to this question is independent of ZFC. More importantly, we introduced in [13] a reflection principle, we called it Fodor-type reflection principle, that is much weaker than Fleissner's Axiom R but still implies most of its known consequences, in particular (the consistency of) the affirmative answer to Arhangelskii's question. The topological methods used to establish this can also be applied under various other circumstances. Thus another interesting result from [13], proved in ZFC, is that metrizability has the singular compactness property in the class of locally separable and countably tight spaces. That is, if every subspace of such a space X of size smaller than $|X|$ is metrizable so is X, provided that $|X|$ is a singular cardinal.

The interest in compact spaces is partly explained by the fact that the Banach spaces $C(K)$ of all continuous functions defined on a compact space K provide many interesting examples in Banach space theory. As an example, by sharpening our earlier result saying that the square of any compactum K contains a discrete subspace of size equal to the density of K, in [20] we obtained the following result that is of interest for functional analysts: Every compactum K possesses a bidiscrete system of size $d(K)$. A bidiscrete system for K is a set of pairs $\{(x_\alpha, y_\alpha) : \alpha < \kappa\} \subset K^2$ such that there are continuous real functions $\{f_\alpha : \alpha < \kappa\} \subset C(K)$ with the property that f_α separates the pair (x_α, y_α) but does not separate any of the other pairs. A bidiscrete system thus provides for the Banach space $C(K)$ a so-called nice biorthogonal system.

The results of [5] are also closely related to compactness. It is well-known that a space is compact if and only if every infinite subset of it has a complete accumulation point. So it is natural to call the space κ-compact if every subset of size κ has a complete accumulation point. Moreover, Arhangelskii calls a space *uncountably compact* if every uncountable subset of it has a complete accumulation point. The main result of [5] is an interpolation theorem for κ-compactness, for singular cardinals κ that implies the following very surprising result: If a space is ρ-compact for every uncountable regular cardinal ρ and is $\aleph_\omega$-compact then it is uncountably compact. It is worth while to mention that the proof of the interpolation theorem uses some deep results of Shelah's PCF theory.

– scattered spaces

We continued the study of cardinal sequences of locally compact scattered spaces, which is a classical but still very active research area in set-theoretic topology.

Every scattered space X can be divided naturally into layers, defined recursively by "$I_\xi(X)$ *is the set of isolated points of the subspace* $X \setminus \bigcup_{\zeta < \xi} I_\zeta(X)$". The supremum of the indexes of non-empty layers is the height of the scattered space.

The sequences of cardinalities of non-empty layers is called the cardinal sequence of X and denoted by $SEQ(X)$.

The general question is the following: *Which functions can be cardinal sequences of compact scattered spaces of height α?* This question has a long history.

Denote $C_\lambda(\delta)$ to denote the set of cardinal sequences of locally compact scattered spaces of height δ that have λ as their starting and minimal value.

In [16] we established the existence of $C_\lambda(\delta)$-universal spaces for various λ and δ. As an application, constructing a suitable universal space we showed the consistency

of $2^\omega = \omega_2$ plus $C_\omega(\omega_2)$ is as large as possible: it consists of all $\{\omega, \omega 1, \omega 2\}$-valued sequences of length ω_2 that start with ω.

In [17] we addresses the following version of the general question: for which sequences f of regular cardinals in a ground model with GCH can we find a cardinal-preserving extension in which GCH holds and f is a cardinal sequence of some LCS space?

Let $\alpha < \lambda^{++}$, Define $\mathcal{D}_\lambda(\alpha) = \{f \in {}^\alpha\{\lambda, \lambda^+\} : f(0) = \lambda, f^{-1}\{\lambda\}$ is $< \lambda$-closed and successor-closed in α $\}$. We showed that for each uncountable regular cardinal λ and ordinal $\alpha < \lambda^{++}$ it is consistent with GCH that $\mathcal{C}_\lambda(\alpha)$ is as large as possible, i.e.

$$\mathcal{C}_\lambda(\alpha) = \mathcal{D}_\lambda(\alpha).$$

Under GCH this gave a consistent characterization of those sequences f or regular cardinals which can be cardinal sequences of some LCS space in some cardinal and GCH preserving extension of the ground model.

By using the combinatorial notion of the *new Δ property (NDP)* of a function, it was proved by Roitman that the existence of an LCS spaces with cardinal sequence $\langle\omega\rangle\frown_{\omega_1} \langle\omega_2\rangle$ is consistent with ZFC Roitman's result was generalized in by Koepke and Martinez, where for every infinite regular cardinal κ, it was proved that the existence of an LCS space with cardinal sequence $\langle\kappa\rangle\frown_{\kappa^+} \langle\kappa^{++}\rangle$ is consistent with ZFC.

In [28] we gave a far-fetching generalization of this results. Especially, we could prove that both the sequence $\langle\omega\rangle\frown_{\omega_1} \langle\omega_3\rangle$ ans the sequence $\langle\omega_1\rangle\frown_{\omega_2} \langle\omega_4\rangle$ can be cardinal sequences of locally compact scattered spaces in a suitable generic extension.

Baumgartner and Shelah proved that it is relatively consistent with ZFC that $\langle\omega\rangle_{\omega_2}$ is a cardinal sequences of locally compact scattered space Refining their argument, first Bagaria, proved that ${}^{\omega_2}\{\omega, \omega_1\} \subset \mathcal{C}(\omega_2)$ in some ZFC model, then we showed that $2^\omega = \omega_2$ and ${}^{\omega_2}\{\omega, \omega_1, \omega_2\} \subset \mathcal{C}(\omega_2)$ is also consistent.

For a long time ω_2 was a mystique barrier in both height and width. In [29] we could construct wider spaces: if GCH holds and $\lambda \geq \omega_2$ is a regular cardinal, then in some cardinal preserving generic extension $2^\omega = \lambda$ and every sequence $\mathbf{s} = \langle s_\alpha : \alpha < \omega_2\rangle$ of infinite cardinals with $s_\alpha \leq \lambda$ is the cardinal sequence of some locally compact scattered space.

We could find the suitable generic extension in three steps: the first extension added a "*strongly stationary strong (ω_1, λ)-semimorass*" to the ground model; using that strong semimorass the second extension added a $\Delta(\omega_2 \times \lambda)$-*function* to the first extension; finally using the $\Delta(\omega_2 \times \lambda)$-function we added an "*LCS space with stem*" to the second model and we showed that those space alone guarantees that every sequence $\mathbf{s} = \langle s_\alpha : \alpha < \omega_2\rangle$ of infinite cardinals with $s_\alpha \leq \lambda$ is the cardinal sequence of some locally compact scattered space.

– resolvability

A topological space X is called κ-resolvable if it contains κ disjoint dense subsets, and maximally resolvable if it is $\Delta(X)$-resolvable where $\Delta(X)$ is the smallest size of a non-empty open set in X. Both metric spaces and linearly ordered spaces are known to be maximally resolvable, and monotonically normal (MN) spaces form a class that includes them both. Thus it seems natural to raise the question if MN spaces are maximally resolvable. We had investigated this problem earlier and found some interesting and unexpected results:

(1) Every dense-in-itself MN space is ω-resolvable.

(2) If κ is a measurable cardinal then there is a MN space X with $\Delta(X) = \kappa$ which is not ω_1-resolvable.

(3) Every MN space of cardinality $< \aleph_\omega$ is maximally resolvable.

(4) From a supercompact cardinal we get the consistency of a MN space X with $|X| = \Delta(X) = \aleph_\omega$ that is not ω_2-resolvable.

The connection of the harmless looking topological problem with some deep set-theory comes from a new class of MN spaces that we called them filtration spaces. These are defined on infinitely branching trees with the help of ultrafilters and the resolvability properties of these spaces depend on the descendingly completeness properties of the ultrafilters used in their construction.

We continued this work in [34] and obtained results that shed new light on the earlier results and completely settled the problems that were left unsolved by them. We showed in [34] that, for every fixed cardinal λ, all MN spaces of cardinality less than λ are maximally resolvable if and only if every uniform ultrafilter on a cardinal less than λ is maximally decomposable. An ultrafilter u on κ is called μ-decomposable if κ can be partitioned into μ sets in such a way that the union of any fewer than μ of them is not in u. Moreover, u is maximally decomposable if it is μ-decomposable for every regular cardinal $\mu \leq \kappa$. By some classical results of Kunen and Prikry, every uniform ultrafilter on a cardinal $< \aleph\omega$ is maximally decomposable, explaining item (3) above. It has been known that the existence of a uniform ultrafilter that is not maximally decomposable implies that a measurable cardinal exists in some inner model, another consequence of this result is the consistency of the maximal resolvability all MN spaces. In fact, it follows that the existence of a monotonically normal space which is not maximally resolvable is actually equiconsistent with the existence of a measurable cardinal.

We could also show in [34], using some results of Woodin, that the consistency of a measurable cardinal implies the existence of an ω_1-irresolvable monotonically normal space X with $|X| = \Delta(X) = \aleph_\omega$. This improves item (4) above to the maximum possible degree, both by replacing the supercompact with a measurable and by replacing ω_2-irresolvability with ω_1-irresolvability.

– D-spaces

A topological space X is a *D-space*, if for every open neighborhood assignment η for X there is a closed discrete subset D of X such that $\eta[D] = X$.

In [7] we showed without using topological games that a space is D if it is a finite union of subparacompact scattered spaces. This result can not be extended to countable unions, since it is known that there is a regular space which is a countable union of paracompact scattered spaces and which is not D. Nevertheless, we showed that every space which is the union of countably many regular Lindelöf $\mathcal{C}$-scattered spaces has the D-property. Also, we prove that a space is D if it is a locally finite union of regular Lindelöf $\mathcal{C}$-scattered spaces.

IV. MODEL THEORY

In [8] we presented a new and short proof for the well known fact, that first order resolution calculus endowed with paramodulation is sound and refutation-complete.

A countable structure A is defined to be absolutely ubiquitous, if for any countable structure B, A and B are isomorphic, whenever the sets of isomorphism types of finite substructures of A and B are the same. Continuing investigations initiated by Hodkinson, Ivanov and others, in [18], we proved that a certain subclass of absolutely ubiquitous structures are $\aleph_0$-stable. This confirms a special case of a conjecture of Macpherson.

Sayed Ahmed recently has shown that there exists an infinite dimensional non-representable Quasi-polyadic Equality Algebra ($QPEA_\omega$, for short) with a representable Cylindric reduct. In [19] we continued related investigations and showed that if $G \subseteq {}^\omega\omega$ is a semigroup containing at least one constant function then a wide class of representable Cylindric Algebras occur as the Cylindric reduct of some non-representable G-PEA_ω. More concretely, we proved that if $\mathcal{A}$ is an ω-dimensional Cylindric Set Algebra with an infinite base set then there exists a non-representable G-PEA_ω whose cylindric reduct is representable and contains an isomorphic copy of $\mathcal{A}$.

In the survey paper [35] we were summing up the developments of the theory of polyadic algebras made in the last two decades.

Let $\mathfrak{c} = 2^{\aleph_0}$. In [33] we gave a family of pairwise incomparable clones on N with $2^\mathfrak{c}$ members, all with the same unary fragment, namely the set of all unary operations. We also gave, for each n, a family of $2^\mathfrak{c}$ clones all with the same n-ary fragment, and all containing the set of all unary operations.

By a celebrated theorem of Morley, a theory T is $\aleph_1$-categorical if and only if it is κ-categorical for all uncountable κ. In [37] we were taking the first steps towards extending Morley's categoricity theorem "to the finite". In more detail, we were presenting conditions, implying that certain finite subsets of certain $\aleph_1$-categorical T have at most one n-element model for each natural number $n \in \omega$ (counting up to isomorphism, of course).

Vaught's Conjecture states, that if T is a complete first order theory in a countable language such that T has uncountably many pairwise non-isomorphic countably infinite models then T has $2^{\aleph_0}$ many pairwise non-isomorphic countably infinite models.

In [36] we prove that if T has at least $\aleph_1$ many countable models which are pairwise separable by critical types, then T has continuum many such models, that is, a certain weak version of Vaught's conjecture is true. The proofs are based on the representation theory of Cylindric Algebras and elementary topological properties of the Stone spaces of these Cylindric Algebras.

Continuing investigations initiated by Sagi, in [38] we applied methods of algebraic logic to study some variants of Vaught's conjecture. More concretely, let $S \subseteq {}^\omega\omega$ be a σ-compact monoid. We proved, among other things, that if a complete first order theory Σ has at least $\aleph_1$ many countable models which cannot be elementarily embedded into each other by elements of S, then, in fact, Σ has continuum many such models. We also study related questions in the context of equality free logics and obtain similar results.

Our proofs were based on the representation theory of cylindric and quasi-polyadic algebras and topological properties of the Stone spaces of these algebras.

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