An investigation of evolutionary potential games using the tensor renormalization group method

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Final report

Balázs Király

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In the present project entitled "An investigation of evolutionary potential games using the tensor renormalization group method" (Grant No. OTKA PD-138571), I set out to apply the tensor renormalization group method introduced by Michael Levin and Cody P. Nave to spin models that bear special relevance to evolutionary game theory.

Recent research has revealed that symmetric matrix games [1] – that is, games in which the payoffs of the two participating players can simply be tabulated according to their choices from a finite set of available strategies and an exchange of the chosen strategies between the two players also leads to an exchange of payoffs – can be linearly decomposed on the analogy of vectors and forces into just four fundamentally different types of elementary game components that describe archetypal interaction situations, namely self-dependent, cross-dependent, coordination-type, and cyclic dominance-type games [2]. Matrix games that lack a cyclic dominance component have the potential game property and thus establish a direct connection between evolutionary game theory and statistical physics: Any repeated potential game governed by the logit strategy update rule is in detailed balance with the Boltzmann distribution defined by its potential, which means that it is equivalent to a classical spin model via correspondence between the available strategies and the spin states, the payoff matrices and the coupling constants, the potential and the negative of the energy.

The tensor renormalization group (TRG) technique introduced by Michael Levin and Cody P. Nave [3] – similarly to a number of related methods [4] – exploits the tensor network structure of the partition function of lattice spin models to evaluate it effectively. On a square lattice with nearest-neighbour interactions, the tensor network structure of the partition function can be made apparent by grouping neighbouring sites into a square lattice of four-site plaquettes. The contraction of the resulting network of four-index tensors can then be carried out systematically through the successive iteration of the following two-step coarse graining transformation: i) Extend the tensor network by splitting the tensors in two via singular value decomposition (SVD) with respect to two different index pairings in a checkerboard pattern. ii) Carry out the original contractions, leaving behind a new square lattice of half as many tensors as the original network composed of summations over the newly introduced SVD indices. Thus, after a suitable number of iterations, any arbitrarily large system can be reduced to a tensor network of manageable size. This, however, does not actually mean that the resulting tensor network is directly calculable, since each coarse graining step doubles the index range of the tensors. Levin and Nave's original, simplest version of the TRG method remedies this situation by applying an interstitial approximation during the SVD step in the form of a truncation that retains only the largest singular values.

As part of the project funded by this grant, I developed my own implementation of the TRG method, which I then used to study two of the model systems that hold the key to understanding the mechanics of the interplay between the elementary games.

The simplest truly interactive matrix game is the elementary coordination game. Its general, n-strategy version consists of two coordinated and $n-2$ neutral strategies: The two players each receive 1 unit of payoff if they both choose the same coordinated strategy, regardless of which one it is; they both lose 1 unit of payoff if they choose opposing coordinated strategies; and neither of them receives any payoff if either one of them chooses one of the neutral strategies. Previous research [5–7] has revealed that the neutral strategies can be bunched together into a single strategy on a regular lattice in a way that is consistent with the logit strategy update rule at the cost of introducing an additional, temperature-dependent self-dependent component that retains the symmetry of the two coordinated strategies. Not only is this an example of the non-trivial ways in which game components can interact in combination, it also establishes another direct connection between evolutionary game theory and statistical physics, as the resulting bunched elementary coordination game can be mapped onto the ferromagnetic Blume–Capel model [8–10], and this remains true even in the presence of a further, constant symmetry-retaining self-dependent component of strength h. The parametrization defined by the mapping traces linear cross sections of the phase diagram of the Blume–Capel model whose slope and intercept are determined by the number of bunched strategies and the strength of the constant symmetry-retaining self-dependent component, respectively.

I applied my implementation of the TRG method to study the square-lattice Blume–Capel model along these unusual – as opposed to the constant crystal field or even the less often considered constant temperature cases – cross sections. My results provide numerical proof of the equivalence of the two models. Measured quantities including the specific heat, the magnetization, its susceptibility and Binder parameter, and the frequency of the neutral strategy/empty sites and its susceptibility indicate the presence of phase transitions at the locations and with the properties found in the literature. (For an illustration, see Figure 1.) When the number of neutral strategies is low and/or the strength of the self-dependent component is high enough, the system exhibits a continuous phase transition that belongs to the Ising universality class. At higher n and/or lower h it has a first-order transition instead. When $h < -0.5$, no phase transition occurs, regardless of n. The continuous and the first-order phase boundaries are separated by a tricritical point, which proved hard to identify on the basis of my TRG results. Nonetheless, the changes in critical behaviour that can be observed in the results still provide a good approximation of the location of the tricritical point.

I extended the above investigation to study the effects of adding a finite magnetic field h' (a self-dependent component that breaks the symmetry of the coordinated strategies) to the Blume–Capel model, which seems to have received less attention in the literature. The results again show good qualitative agreement with the mostly mean-field approximated previous findings (as illustrated by Figure 1), but they are expected to perform better in terms of quantitative detail due to the inherently higher accuracy of the TRG method. Indeed, the observed power law behaviour of the magnetization as a function of the magnetic field at fixed zero-field continuous phase transition critical temperatures and crystal fields seems consistent with the transitions belonging to the Ising universality class. Unlike the continuous zero-field transitions, their first-order counterparts are not immediately smoothed out by any finite magnetic field: Below a critical magnetic field strength, they are just shifted towards a lower critical temperature and a higher critical crystal field; at the critical h', the transition becomes continuous; above it, no phase transition occurs. The resulting finite magnetic field phase boundary can be described as two symmetric (because of the symmetry of the two coordinated strategies) slanted wings attached to the zero-field first-order transition line and bordered from the high-

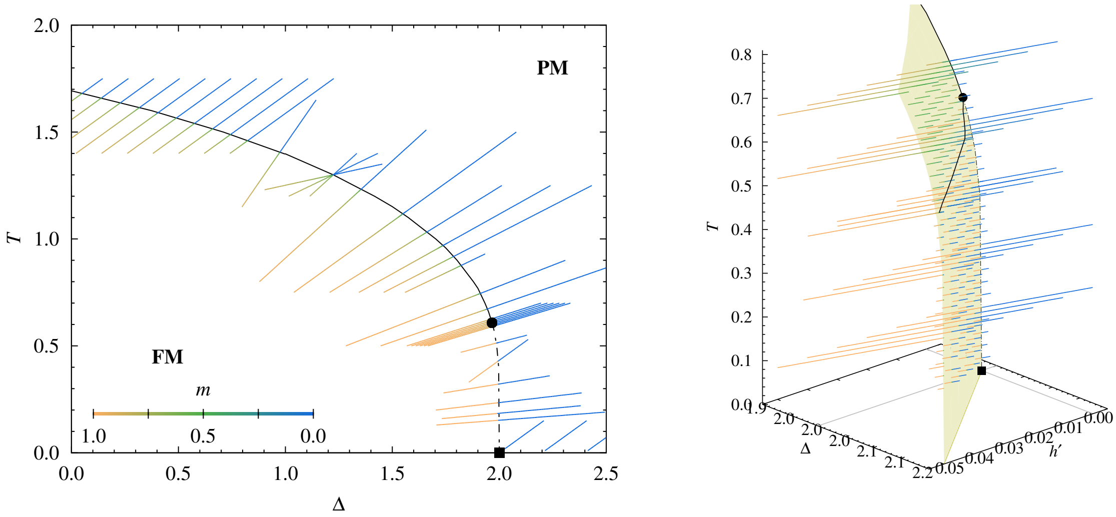

Figure 1: Heat maps of the TRG magnetization along elementary coordination game cross sections of the Blume–Capel model in the absence (left) and presence (right) of an external magnetic field/self-dependent component h'. Δ and T denote the strength of the crystal field and the temperature, respectively. The solid and dash-dotted lines indicate continuous and first-order phase transition boundaries between the ferromagnetic (FM) and paramagnetic (PM) phases. The data used to draw the zero-field phase boundary were taken from Ref. [10], which compiles results from other articles listed therein. The $h' > 0$ solid black curve starting from the tricritical point (black dot) is a rough estimate of the continuous-transition upper boundary of the first-order-transition wing based on TRG results. The yellow surface shows the linear mapping of the zero-field phase boundary that seems to well approximate the location of the $h' \neq 0$ first-order transitions.

temperature side by continuous transition lines that meet in the zero-field tricritical point. The shift in the critical crystal field of low-temperature first-order transitions appears to be approximately proportional to the magnetic field. It was again difficult to exactly locate where first-order transitions become continuous, but the rough estimate extracted from the TRG data does both match the zero-field phase diagram and agree with predictions found in prior studies [11].

Overall, these TRG results corroborate and add to the literature of the Blume–Capel model and, in doing so, demonstrate both that the bunched elementary coordination game and the Blume–Capel model are indeed equivalent to each other and that the TRG method can indeed be effectively used to investigate the behaviour in general and the phase transitions in particular of similar evolutionary game models. I have presented these findings on a poster at the International Conference on Statistical Physics SigmaPhi 2023. A manuscript reporting the above outlined analysis is currently under review at Physica A [P1].

As a next step, I tried applying the same TRG method to a game with a more complex elementary game composition. The (five-strategy) game of competing Ising- and (three-strategy) Potts-type subgames consists of a total of four elementary coordination games, one of unit strength involving the first two strategies and three of equal $\alpha/2$ strength connecting each pair among the last three strategies. Previous investigations [12] revealed using mean-field approximation and Monte Carlo simulations that when this model is played by nearest neighbours located at the sites of a square lattice according to the logit strategy update rule, then it does indeed describe a sort of competition between the two subgames: When α is low ($\alpha \leq 1$), the composite game is dominated by the Ising-type coordination between the first two strategies and the system exhibits a single continuous phase transition that belongs to the Ising universality class; whereas for large α, the only phase transition occurring in the system spontaneously breaks the symmetry of an entirely different order parameter and is characterized by Potts-class critical exponents instead. For values of α in between, both the Ising- and Potts-ordered phases can be observed, separated by a first-order phase transition. In this case the system also has a second, continuous Ising-type order to disorder phase transition at a higher critical temperature. Seemingly, the symmetries of the two subgames are never broken simultaneously, only one at a time.

The TRG method turned out to be much more prone to numerical instability in the case of this model, especially in the intermediate-alpha region, which has hindered the collection of data and, consequently, the publication of results. It is possible that this issue is related to the increased number of available strategies. The resulting higher initial tensor dimension means that setting the same cut-off dimension while truncating the SVD increases the approximation error. Increasing the cut-off dimension and the number of iterations, however, increases the hardware and time requirements for carrying out the calculations, which eventually ran up against the limitations of the project, so a compromise had to be made between numerical stability and precision on the one hand and the number of data points that could be acquired on the other. The instability issues also raise the possibility of the TRG method – at least in its simplest form used throughout this project – not being well suited to the investigation of models with multiple competing order parameters. We can see hints of something similar happening in Monte Carlo simulations of this model, too, where efficiency is significantly decreased by transient domain growth processes. This issue can be mitigated by using prepared initial states, but it also allows the simulation to get stuck in metastable states for long periods of time, which prevents the identification of the actual stable equilibrium state. Maybe similar small modifications or the use of more advanced tensor network techniques that correct some of the inherent shortcomings of the TRG method (e.g., TNR [13] or GILT [14]) can improve or complement the TRG method to give more reliable access to the intermediate-α region of the game of competing Ising- and Potts-type subgames. However, pursuing such investigations ultimately falls beyond the scope and limitations of the present project.

Difficulties notwithstanding, the TRG method still produces new evidence in support of earlier findings about the system's behaviour. This includes examples of single Ising-type, single Potts-type, and consecutive Potts order to Ising order and Ising order to disorder phase transitions showing up in results for the specific heat, Ising and Potts magnetization, and their Binder parameters. (For an illustration, see Figure 2.) The collection of further supporting data is still in progress. A manuscript communicating the results is already in preparation [P2].

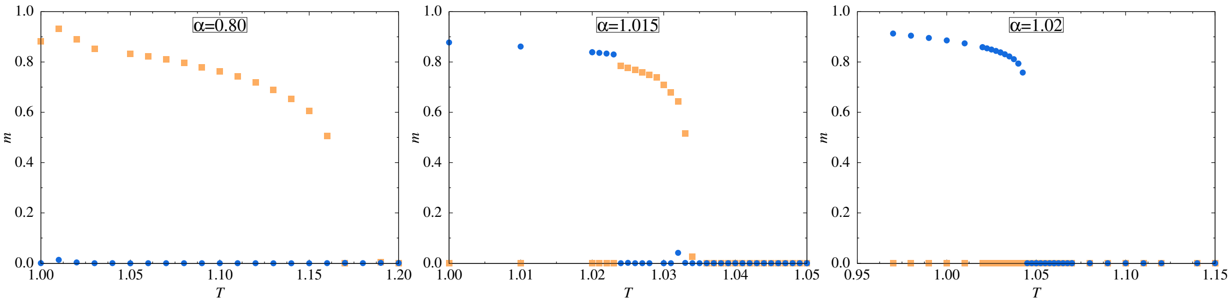

Figure 2: TRG results for the magnetizations of the Ising (orange squares) and Potts (blue circles) subgames for $\alpha = 0.80$, 1.015, and 1.02 (from left to right) in the game of competing Ising- and Potts-type subgames, as a function of the temperature.

Inspired by the bunching of the neutral strategies of the elementary coordination game, I tried to apply the same idea to the game of competing Ising and Potts subgames in the hope of effectively reducing the number of its available strategies and thus improving its numerical stability under the TRG method. Despite the fact that the symmetry of Potts strategies spontaneously breaks in a way that only one of them has a higher frequency than the others, a consistent bunching of the minority Potts strategies turns out to be impossible by way of simply adding temperature-dependent terms to the payoffs of a similar four-strategy game, due to the Potts subgame having different diagonal and off-diagonal elements. However, this analysis does not in itself rule out the possibility of finding such a fewer-strategy or otherwise simpler equivalent by some other means that can better exploit assumptions about how subgame symmetries break in the corresponding ordered phase. Not only could such effective replacement models improve the efficacy of both numerical and analytic investigation techniques in specific cases, they could also provide access to the systematic study of related many-strategy models – research into their existence would make for a worthwhile future project.

In order to better understand the game of competing Ising and Potts subgames and further explore the possibility of finding a simpler equivalent model, I also reexamined its mean-field approximation. More precisely, I looked at the mean-field approximation of a generalized version of the game that combines a p-state and a q-state Potts subgame. The game of competing Ising and Potts subgames is the $p = 2$, $q = 3$ special case of this general $(p+q)$-strategy model. It also contains the p-state (q-state) Potts model itself as its $q = 0$ ($p = 0$) limit. Taking into account the way the subgame symmetries break, I managed to exactly derive a single, one-variable equation whose solution determines both the temperature and the magnetization at the order–disorder transition. The closed-form exact solution of this equation is known in the Potts model limit [15], which raises the possibility of finding an analytic solution for at least some other special – or maybe even the general – cases, too.

While being supported by the present grant, I also helped conduct two further studies that are tangentially related to its topic. My contribution to these collaborations has taken away neither research time nor other resources from this project, but it also would not have been possible without the background provided by the received funding.

With István Borsos (HUN-REN EK) and György Szabó (HUN-REN EK and HUN-REN ÖK), we applied the matrix decomposition scheme that forms one of the bases of the present project to the problem of graph characterization. More specifically, we derived multiple local parameters and global measures from the decomposition of the adjacency matrices of directed graphs and with their help quantified certain topological features of recursive trees. Our results were published (with acknowledgement of the funding I have received) in an article titled "Quantification and statistical analysis of topological features of recursive trees" in Physica A [P3].

Classical matrix game models of evolutionary processes consider games defined by a constant payoff matrix. With Tamás Varga (University of Szeged), György Szabó (HUN-REN EK and HUN-REN ÖK), and József Garay (HUN-REN ÖK), we explored – using the example of a modified version of the hawk–dove game – how the notion of evolutionary stability extends to games in which not only the players' strategy choices but also the payoffs describing their interactions are determined by independent traits that evolve according to natural selection. A manuscript detailing our findings (and acknowledging the funding I have received) is currently under review at the Journal of Mathematical Biology [P4].

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Manuscripts and articles related to the project

[P1] B. Király, "A tensor renormalization group analysis of the Blume–Capel model inspired by game theory," under review at Physica A (2023).

[P2] B. Király, "A tensor renormalization group analysis of an evolutionary game of competing Ising and Potts subgames," in preparation (2023).

[P3] B. Király, I. Borsos, and G. Szabó, "Quantification and statistical analysis of topological features of recursive trees," Physica A **617**, 128672 (2023).

[P4] B. Király, T. Varga, G. Szabó, and J. Garay, "Evolutionarily stable payoff matrix in hawk–dove games," under review at J. Math. Biol. (2023).