Final report of NKFI FK134251

Dimension Theory of Iterated Function Systems

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This document describes the research activities of the grant NKFI FK134251 at the Department of Stochastics of the Mathematics Institute of the Budapest University of Technology and Economics (BUTE) from 2020.09.01 to 2024.08.31.

General overview

The first two years of the application were significantly affected by the COVID-19 pandemic. During that period, travelling and attending workshops were not possible, and the availability of appropriate IT tools became extremely important. Thus, compared to our original financial plan, we carried out significant reallocations from the material costs to capital goods so that research could proceed at an appropriate pace and scientific collaborations would remain smooth. In September 2021, Vilma Orgoványi (PhD student of Károly Simon) joined to the PI (Balázs Bárány) in the project in the position "Researcher to be employed" of the application. Her task was to study the geometric properties of random fractals.

After the pandemic, we organized the workshop "Geometry of Deterministic and Random Fractals" from 27th June to 1st July 2022 at the Budapest University of Technology and Economics, honouring the 60+1th birthday of Károly Simon. It was a great opportunity to discuss new research directions, exchange ideas, and establish new scientific cooperation. Without this grant and the support of the NKFI, this would not have been possible, as the application covered the accommodation of the invited speakers. For further details, see `simon60.math.bme.hu`.

In 2023, the PI together with Károly Simon and Boris Solomyak published a book *Self-similar and Self-affine Sets and Measures* [18]. The book contains both introductory material for beginners and more advanced topics in the theory of self-similar and self-affine systems, which continue to be the focus of active research. Among the latter are self-similar sets and measures with overlaps, including the much-studied Bernoulli-convolutions. Self-affine systems pose additional challenges. Their study is often based on ergodic theory and dynamical systems methods. In the last twenty years, there have been many breakthroughs in these fields. The aim of the book [18] is to give introduction to some of them, often in the simplest non-trivial cases. The book is intended for a wide audience of mathematicians interested in fractal geometry, including students. Parts of the book can be used for graduate and even advanced undergraduate courses.

On top of that, the members of the group, the PI and Vilma Orgoványi, together with their co-authors have published 19 articles during the grant period. Out of the 19 articles, seven have already been published in the journals *Acta Arithmetica, Advances in Mathematics, Discrete & Continuous Dynamical Systems, International Mathematics Research Notices, Nonlinearity, Studia Mathematica* and *The Asian Journal of Mathematics*; four have been accepted for publication or have appeared online in journals *Communications in Mathematical Physics, Ergodic Theory & Dynamical Systems, Proceedings of the American Mathematical Society* and *Proceedings of the Royal Society of Edinburgh Section A*.

Our research focused on several aspects of the theory of iterated function systems: We have studied the dimension theory of self-conformal and non-conformal systems; absolute continuity and dimension of stationary measures; finer geometric properties, like Hausdorff measure and slices of non-conformal systems; dynamically defined subsets; and randomly constructed conformal sets. Furthermore, we have successfully found applications of the theory of iterated functions systems in real analysis, number theory and even collisional abrasion models. In the rest of the report, we describe the results achieved in the published articles, grouped by topic.

I. Self-conformal Iterated Function Systems

● *Typical absolute continuity for classes of dynamically defined measures* [19]

● *Typical dimension and absolute continuity for classes of dynamically defined measures, part II: Exposition and Extensions* [20]

Let $\{f_i^\lambda\}_{i=1}^m$ be a parametrised family of smooth uniformly contractive iterated function systems (IFS) on the real line with parameter $\lambda \in U$. In [19], we considered one-parameter families, which we extended to the multiparameter case in [20]. Given a family of parameter-dependent measures μ_λ on the symbolic space $\Sigma = \{1, \ldots, m\}^{\mathbb{N}}$, we studied geometric and dimensional properties of their images $(\pi_\lambda)_*\mu_\lambda$ under the natural projection maps $\pi_\lambda : \Sigma \to \mathbb{R}$ defined as $\pi_\lambda(i_1, i_2, \ldots) = \lim_{n\to\infty} f_{i_1}^\lambda \circ \cdots f_{i_n}^\lambda(0)$. The main novelty of our work is that the measures depend on the parameter. In contrast, up till now, it has been usually assumed that the measure on the symbolic space is fixed, and the parameter dependence comes only from the natural projection, see for example Simon, Solomyak and Urbański [43]. However, families of parameter-dependent measures arise naturally, in particular, invariant measures for IFSs with place-dependent probabilities and natural (equilibrium) measures for smooth IFSs.

Our main result states that if the parameter-dependent family of measures μ_λ are Gibbs measures for a family of Hölder continuous potentials $\phi^\lambda : \Sigma \to \mathbb{R}$, with Hölder continuous dependence on λ and the parametrised family of IFSs satisfy the transversality condition, then the projected measure is absolutely continuous for Lebesgue a.e. parameters λ for which the ratio of entropy over the Lyapunov exponent is strictly greater than 1. We deduced it from a more general, almost sure lower bound on the Sobolev dimension for families of measures with regular enough dependence on the parameter. Furthermore, we obtain formulas for the Hausdorff dimension for almost every parameter.

As applications of our results, we study stationary measures for iterated function systems with place-dependent probabilities (place-dependent Bernoulli convolutions and the Blackwell measure for binary channel) and equilibrium measures for hyperbolic IFS with overlaps (in particular, natural measures for non-homogeneous self-similar IFS and certain systems corresponding to random continued fractions).

The paper [20] is also partly an exposition based on the series of lectures by Károly Simon at the Summer School "Dynamics and Fractals" in 2023 at the Banach Center, Warsaw. One of the goals of that paper is to present an exposition of [19] in a more reader-friendly way, emphasising the ideas and proof strategies but omitting the more technical parts.

● *Scaling limits of self-conformal measures* [13]

In this paper, we studied the uniformly scaling property of self-conformal measures. We call a measure μ on $\mathbb{R}^d$ self-conformal if there exists an IFS $\{f_i\}_{i=1}^m$ of smooth contractive conformal mappings on $\mathbb{R}^d$ and a probability vector $(p_i)_{i=1}^m$ such that $\mu = \sum_{i=1}^m p_i (f_i)_*\mu$. If the IFS consists of similarities, we say the measure μ is self-similar.

The notion of uniform scaling appeared first in a work of Gavish [29], where he also showed that self-similar measures on $\mathbb{R}^d$ with the open set condition (that is, there exists an open and bounded set $U \subset \mathbb{R}^d$ such that $f_i(U) \subset U$ and $f_i(U) \cap f_j(U) = \emptyset$ for every $i \neq j$) are uniformly scaling. A more direct ergodic-theoretic proof was introduced by Hochman [31] in his systematic study of tangent distributions. Later, Hochman and Shmerkin [34] showed that self-conformal measures on $\mathbb{R}$ with the open set condition are uniformly scaling. In the main result of the manuscript, we show that any self-conformal measure on $\mathbb{R}$ and any self-similar measure on $\mathbb{R}^d$ is uniformly scaling and generates an ergodic fractal distribution. This generalises existing results by removing the need for any separation condition. As a consequence, we obtain applications to the prevalence of normal numbers in self-conformal sets and to the dimensions of convolutions of self-conformal measures on the line, both of which are active research topics on their own, see, for example, Algom, Baker and Shmerkin [1] and Hochman and Shmerkin [33].

● *Covering number on inhomogeneous graph-directed self-similar sets* [12]

Mauldin and Williams [38] introduced the concept of graph-directed self-similar sets, and later, Dubey and Verma [23] introduced the so-called inhomogeneous graph-directed self-similar sets. Hambly and Nyberg [30] studied the asymptotic behaviour of the covering number $N_r(A) = \min\{k \in \mathbb{N} : A \subset \bigcup_{i=1}^k B(x_i, r)\}$ for graph-directed self-similar sets under strong open set condition. In this manuscript, we completely characterised the asymptotic behaviour of the covering number for strongly connected inhomogeneous graph-directed self-similar sets satisfying the strong open set condition.

II. Non-conformal Iterated Function Systems

● *Finer geometry of planar self-affine sets* [14]

Let $\{f_i(x) = A_i x + t_i\}_{i=1}^m$ be a planar IFS consisting of invertible affine contractions. We call the unique non-empty compact set X satisfying $X = \bigcup_{i=1}^m f_i(X)$ self-affine set. For planar self-affine set satisfying the strong open set condition and strong irreducibility (i.e. there is no finite collection of proper subspaces preserved by all of the matrices A_i), it has been recently proved by Bárány, Hochman and Rapaport [9] that the Hausdorff dimension equals the affinity dimension. In this article, we continued this line of research, and our objective was to acquire more refined geometric information. In our first main result, we have determined the lower dimension of planar self-affine sets satisfying the strong irreducibility (SI) and strong separation condition (SSC), that is, $f_i(X) \cap f_j(X) = \emptyset$ for $i \neq j$). Consequently, we have shown that if the Hausdorff dimension is strictly greater than 1, the planar self-affine set with SI and SSC cannot be Ahlfors regular.

Assuming domination (that is, there exists a closed multicone as a proper subset of the projective circle mapped into the interior of itself by all of the matrices) in addition to the condition of strong separation and strong irreducibility, we gave a characterisation of the Ahlfors regularity of the self-affine set. In particular, for a self-affine set with SSC, SI and domination, the following are equivalent:

- the self-affine set is Ahlfors regular,
- the Hausdorff dimension is smaller than or equal to 1, and its proper-dimensional Hausdorff measure is positive,
- the IFS satisfies a certain projective separation condition, which is a natural adaptation of the condition given by Bandt and Graf [6] for self-similar sets.

Furthermore, we have shown that if the projective separation fails, then the Assouad dimension of the set is at least 1.

Finally, under the conditions above, we have determined the Assouad dimension of the self-affine sets with Hausdorff dimension greater than or equal to one, and we have shown that it is one plus the maximum of the Hausdorff dimension of the line segments of X. Using that, we answered a question posed by Fraser [28, Question 17.5.2], and we gave an example of a self-affine set with equal Hausdorff and affinity dimensions but strictly different Assouad and lower dimensions.

● *Slices of the Takagi function* [4]

In this paper, we continued our research along the lines of [14] and generalised the results regarding the Assouad dimension. We have shown that for any planar self-affine set satisfying the bounded neighbourhood condition (BNC), the Assouad dimension is bounded above by the maximum of the Hausdorff dimension of the set and the supremum of the Hausdorff dimension of the line segments plus one. Furthermore, we showed equality if we assume domination and that the dimension of the orthogonal projection in any direction is one. This result is a generalisation of the previous result in [14] since under the SSC and SI, the dimension of the projection of the self-affine set is one in every direction, see Bárány, Hochman and Rapaport [9].

We applied our results to the Takagi function, a nowhere differentiable real function with a self-affine graph. We have also proved that the Assouad dimension equals the affinity dimension if and only if the upper pointwise dimension of every projection of the occupation measure (the Lebesgue measure on the x-axis lifted to the graph) at every point is at least one.

● *On the dimension of planar self-affine sets with non-invertible maps* [15]

In this paper, we continued to study the geometric properties of planar self-affine sets and extend the result of Bárány, Hochman and Rapaport [9] for planar self-affine sets, of which generating IFS contains non-invertible affine mappings. The case when the affinity dimension is strictly greater than one was handled by Käenmäki and Nissinen [35], and our result completes their result. We showed that if the affinity dimension is smaller than or equal to one, then under a certain separation condition, the dimension equals the affinity dimension for a typical choice of the linear parts of the non-invertible mappings. Furthermore, we show that if the affinity dimension is strictly smaller than one, then the dimension drops (i.e. it is strictly smaller than the affinity dimension) for a relatively large set of parameters.

● *Dimension of planar non-conformal attractors with triangular derivative matrices* [11]

Thanks to the recent results of Bárány, Hochman, and Rapaport [9] and Hochman and Rapaport [32], the dimension theory of planar self-affine sets is now much better understood. However, the dimension theory of the attractors and invariant measures of iterated function systems consisting of non-conformal and non-linear maps is far from being well understood. This paper was devoted to taking a step forward in this direction.

We studied the dimension of the attractor and quasi-Bernoulli measures of parametrised families of iterated function systems of non-conformal with triangular derivative matrices. We introduced a transversality condition under which, relying on a weak Ledrappier-Young formula, we showed that the dimensions equal to the root of the subadditive pressure and the Lyapunov dimension, respectively, for almost every choice of parameters. We also exhibit concrete examples satisfying the transversality condition concerning the translation parameters.

We have to note here that Feng and Simon [27] recently established another transversality condition for non-linear, non-conformal maps independently. However, the presented examples by Feng and Simon [27] have significantly different structure than the systems studied in [11].

III. Multifractal Analysis and Chaos Game

● *On the convergence rate of the chaos game* [10]

In this paper, we studied how long it takes the orbit of the chaos game to reach a certain density inside the attractor X of an IFS $\{f_i\}_{i=1}^m$ formed by strict contractions on $\mathbb{R}^d$, and we only assume that the lower dimension of the attractor is positive. For an $\mathbf{i} = (i_1, i_2, \dots) \in \Sigma = \{1, \dots, m\}^{\mathbb{N}}$, let us define the orbit of the chaos game driven by $\mathbf{i}$ by $O_n(\mathbf{i}, x) = \{x, f_{i_1}(x), \cdots, f_{i_n} \circ \cdots \circ f_{i_1}(x)\}$. Let μ be a left-shift invariant ergodic probability measure on Σ with exponential decay of correlation.

We showed that the μ-almost sure rate of the growth of the cover time $T_r(\mathbf{i}, x) = \inf\{n \geq 1 : d_H(X, O_n(\mathbf{i}, x)) < r\}$ as $r \to 0$ (where $d_H(.,.)$ denotes the Hausdorff distance of compact sets) is determined by the Minkowski dimension of the push-forward $\pi_*\mu$ of μ. This dimension concept was recently introduced by Falconer, Fraser and Käenmäki [24]. Our result generalised significantly the previous result on the chaos game by Morris and Jurga [39].

Moreover, we bound the expected cover time value from above and below with multiplicative logarithmic correction terms. As an application, we completely characterised the family of probability vectors for Bedford-McMullen carpets, which minimises the Minkowski dimension of Bernoulli measures. Interestingly, these vectors have not appeared in any other aspect of Bedford-McMullen carpets before.

● *Dynamically defined subsets of generic self-affine sets* [21]

In dynamical systems, shrinking target sets and pointwise recurrent sets are two important classes of dynamically defined subsets. In this paper, we studied the cylindrical shrinking target and recurrence sets over self-affine sets in $\mathbb{R}^d$. That is, for a given IFS $\{f_i(x) = A_i x + t_i\}_{i=1}^m$ of affinities and sequence of finite words $\lambda_k = (\lambda_{k,1}, \dots, \lambda_{k,n_k})$ formed by the symbols $1, \dots, m$, let $S((\lambda_k)_k) = \{(i_1, i_2, \dots) \in \Sigma : (i_{k+1}, \dots, i_{k+n_k}) = \lambda_k$ for infinitely many $k \in \mathbb{N}\}$. One can define the cylindrical recurrence sets similarly. That is, for a given $\psi\colon \mathbb{N} \to \mathbb{N}$, let

$$R(\psi) = \{(i_1, i_2, \dots) \in \Sigma : (i_{k+1}, \dots, i_{k+\psi(k)}) = (i_1, \dots, i_{\psi(k)}) \text{ for infinitely many } k \in \mathbb{N}\} \quad (1)$$

.

We introduced a mild irreducibility-like condition on the linear parts $\{A_1, \dots, A_m\}$ that allow us to bound the Hausdorff dimension of the cylindrical shrinking target set $\pi(S((\lambda)_k))$ and recurrence set $\pi(R(\psi))$ using the so-called subadditive pressure function, where $\pi\colon \Sigma \to \mathbb{R}^d$ is the natural projection. We proved that these bounds are sharp for self-affine sets with Lebesgue typical translation parameters and $\|A_i\| < 1/2$ for every i. These mild assumptions mean that our results significantly extend and complement the existing literature for recurrence on self-affine sets, see Koivusalo and Ramírez [36].

● *Recurrence rates for shifts of finite type* [2]

It is clear that the symbolic space and the left-shift dynamics plays a crucial role in the theory of iterated function systems. In this paper and the next two papers, we will discuss the result we achieved for general shift dynamics. Let A be an $m \times m$ matrix with entries $0, 1$. Let $\Sigma_A = \{(i_1, i_2, \dots) \in \{1, \dots, m\}^{\mathbb{N}} : A_{i_1, i_2} = 1\}$ be a sub-shift of finite type. Suppose that it is topologically mixing, which is equivalent to the matrix A being irreducible and aperiodic. Let $\sigma\colon \Sigma_A \to \Sigma_A$ be the usual left-shift, and let μ be a Gibbs measure for a Hölder continuous potential that is not cohomologous to a constant.

In this paper, we studied recurrence rates for the dynamical system (Σ_A, σ) that holds μ-almost surely. In particular, given a function $\psi\colon \mathbb{N} \to \mathbb{N}$, we are interested in the recurrence set $R(\psi)$, defined in (1). We provide sufficient conditions for $\mu(R(\psi)) = 1$ and sufficient conditions for $\mu(R(\psi)) = 0$. As a corollary of these results, we discover a new critical threshold where the measure of $R(\psi)$ transitions from zero to one. This threshold was previously unknown, even in the special case of a non-uniform Bernoulli measure defined on the full shift. The proofs of our results combine ideas from Probability Theory and Thermodynamic Formalism. We also applied our results to the study of dynamics on self-similar sets.

● *On the multifractal spectrum of weighted Birkhoff averages* [16]

In this paper, we studied the topological entropy spectrum h_{top} of weighted Birkhoff averages over topologically mixing sub-shifts of finite type Σ_A. More precisely, let $\phi\colon \Sigma_A \to \mathbb{R}$ be a continuous potential. The topological entropy spectrum of the Birkhoff averages, i.e. the map $\alpha \mapsto h_{\text{top}}\left(\left\{\mathbf{i} \in \Sigma_A : \lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1}\phi(\sigma^k\mathbf{i}) = \alpha\right\}\right)$, has been widely studied, see for example Barreira, Saussol and Schmeling [22]. However, we have limited knowledge if one considers weighted Birkhoff averages, namely, for a sequence of non-negative weights $(w_k)_{k=0}^{\infty}$, the spectrum $\alpha \mapsto h_{\text{top}}\left(\left\{\mathbf{i} \in \Sigma_A : \lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} w_k\phi(\sigma^k\mathbf{i}) = \alpha\right\}\right)$.

We showed that if the sequence of weights is bounded, then the spectrum is continuous and concave over its domain. In the case of typical weights concerning some ergodic quasi-Bernoulli measure ν, i.e. for ν-almost every weight $(w_k) \in \{1, \dots, N\}^{\mathbb{N}}$ with some $N \geq 2$, we determined the spectrum explicitly by proving a conditional variational principle and thermodynamic formalism. Moreover, in the case of full shift ($\Sigma_A = \Sigma$) and under the assumption that the potentials depend only on the first coordinate (i.e. $\phi(\mathbf{i}) = \phi_{i_1}$), we showed that our result is applicable for regular weights (i.e. $\lim_{n\to\infty} \frac{1}{n}\#\{0 \leq k \leq n : w_k = \ell\}$ exists for every $\ell = 1, \dots, N$), like the Möbius sequence. Fan [25] studied similar problems in his paper with strictly different methods.

● *Spectrum of weighted Birkhoff average* [17]

In this paper, we studied the theory of weighted Birkhoff averages from a different perspective. Let $\{s_n\}_{n\in\mathbb{N}}$ be a decreasing non-summable sequence of positive reals. We studied the weighted Birkhoff average $\frac{1}{S_n}\sum_{k=0}^{n-1} s_k\phi(\sigma^k\mathbf{i})$ on topologically mixing sub-shift of finite type Σ_A, where $\phi\colon \Sigma_A \to \mathbb{R}$ is a continuous potential and $S_n = \sum_{k=0}^{n} s_k$. Firstly, we showed that the topological entropy spectrum of that type of weighed Birkhoff averages remains the same as the spectrum of the Birkhoff averages. That is,

$$h_{\text{top}}\left(\left\{\mathbf{i} \in \Sigma_A : \lim_{n\to\infty} \frac{1}{S_n}\sum_{k=0}^{n-1} s_k\phi(\sigma^k\mathbf{i}) = \alpha\right\}\right) = h_{\text{top}}\left(\left\{\mathbf{i} \in \Sigma_A : \lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1}\phi(\sigma^k\mathbf{i}) = \alpha\right\}\right) \text{ for every } \alpha \in \mathbb{R}.$$

Then we calculated the packing spectrum $\dim_P$ of the weighed Birkhoff averages. It turned out that we can have two cases: either the packing spectrum equals to the topological entropy spectrum or for every non-empty level set, the packing spectrum is the constant function equals to the dimension of the whole space over its domain. We have characterised these two cases.

IV. Random Iterated Function Systems

● *Projections of the random Menger sponge* [40]

The deterministic Menger sponge is a well-known fractal set, which is a 3-dimensional analogue of the Sierpiński carpet. Using a similar random process to the one which yields the fractal percolation sets, starting from the deterministic Menger sponge we attain the random Menger sponge (which depends on a probability parameter $p \in (0, 1)$). In this paper, we studied the projections of this set along the direction given by the diagonal of the unit cube. Namely, we studied for which parameters p the projection has a positive Lebesgue measure and for which parameters the projection has a non-empty interior almost surely conditioned on non-extinction. It is an open question whether a deterministic self-similar set exists on the line with positive Lebesgue measure and empty interior. The orthogonal projection of the random Menger sponge to the diagonal is a random self-similar set on the line, and there exists a parameter interval of probabilities where the Lebesgue measure of the projection is positive for almost every non-empty realisation, but the set has empty interior for almost all realisations. In this paper, we further considered a family of random self-similar sets on the line, which are, in some sense, the generalisations of rational projections of the random Menger sponge. For this more general family of sets, we studied what easy-to-check conditions we can give to guarantee the existence of interior points and the positivity of the Lebesgue measure.

● *Interior points and Lebesgue measure of overlapping Mandelbrot percolation sets* [41]

In some sense, this paper is the continuation of [40]. In this paper, we considered the same family of random self-similar sets, but now in higher dimensions, not just on the line. In this paper, we gave sharp but (most of the time) impossible-to-check conditions under which such sets have positive/zero Lebesgue measure. Further, in some special cases, we gave sharp conditions for the existence of interior points. Lastly, we investigated when we can almost surely guarantee the existence of an interval of probability parameters where the set has a positive Lebesgue measure and an empty interior.

● *Multitype branching processes in random environments with not strictly positive expectation matrices* [42]

In this paper, we investigated the survival probability of multitype branching processes in random environments with not strictly positive expectation matrices. It is well known that when the expectation matrices are all strictly positive, the positivity of the survival probability depends on the positivity of the Lyapunov exponents of the expectation matrices. For the use of this result in the study of some random fractals, it is necessary to relax this positivity condition. In particular, in this paper, we proved that the statement holds under the following relaxed conditions. First, there exists a product of the expectation matrices which is strictly positive and second, every expectation matrix contains a positive element in every row and every column.

V. Applications

● *On an abrasion motivated fractal model* [7]

In this paper, we considered a fractal model motivated by the abrasion of convex polyhedra, where the abrasion is realised by chipping small neighbourhoods of vertices. We provided a formal description of the successive chippings using non-autonomous iterated function systems formed by affine maps, and we showed that the net of edges converges to a compact limit set under mild assumptions. We studied the upper box-counting dimension of the limiting object, and we gave a formula for it in the form of a natural modification of the affinity dimension for non-autonomous affine IFSs. Furthermore, we studied the Hausdorff dimension of the limiting object of the net of edges after infinitely many chipping in the case when the chipping rates are constant, in which case the limiting object is a finite union affine images of actual self-affine sets in $\mathbb{R}^3$.

● *Level sets of prevalent Hölder functions* [3]

Adapting the so-called transversality method from the theory parametrised IFSs and using Fourier analytic methods, we studied the level sets of prevalent Hölder functions. For a prevalent α-Hölder function on the unit interval, we showed that the upper Minkowski dimension of every level set is bounded from above by $1-\alpha$ and Lebesgue positively many level sets have Hausdorff dimension equal to $1-\alpha$. This partially answers the question of Balka, Darji and Elekes [5, Problem 9.2].

● *Lagrange-like spectrum of perfect additive complements* [8]

Finally, in this paper we considered a number theoretic application of iterated functions systems. Two infinite subsets A and B of non-negative integers are called perfect additive complements of non-negative integers if every non-negative integer can be uniquely expressed as the sum of elements from A and B. Fang and Sándor [26] characterised the structure of perfect additive complements and established a one to one correspondence between perfectly additive complements A and B and infinite sequences $(m_k) \in \{2,3,\ldots\}^{\mathbb{N}}$. Ma [37] showed that $\limsup_{k\to\infty} \frac{1}{k}\#(A\cap[1,k])\#(B\cap[1,k]) = \limsup_{k\to\infty} G_{m_k}\circ\cdots\circ G_{m_1}(2)$, where (m_k) is the corresponding infinite sequence and $G_m(x) = \frac{2mx}{(m+2)x-2}$. That is, $\limsup_{k\to\infty} \frac{1}{k}\#(A\cap[1,k])\#(B\cap[1,k])$ is the largest accumulation point of the orbit of the chaos game driven by the sequence (m_k) with respect to the infinite IFS $\Phi = \{G_m(x)\}_{m=2}^{\infty}$.

In this paper, we studied the Lagrange-like spectrum $\{\limsup_{k\to\infty} G_{m_k}\circ\cdots\circ G_{m_1}(2) : (m_k)\in\{2,3,\ldots\}^{\mathbb{N}}\}$. We obtained its smallest accumulation point and the discrete part of the set before the smallest accumulation point. Furthermore, we showed that the Lagrange-like spectrum is closed and it contains an interval. This shows several properties similar to the usual Lagrange spectrum of Diophantine approximation and significantly extends the previously known results.

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