Delay Differential Equations
K 129322
Project closure report
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1. The main goals

Delay differential equations (DDEs) are used to model phenomena where the rate of change of the system state at a given time depends on past states, i.e., on the history of the system. There is a wide range of applications in physics, chemistry, biology, engineering, and social sciences as well. DDEs form a particular class of infinite dimensional dynamical systems.

An autonomous nonlinear DDE has the form

$$\dot{x}(t) = F(x_t) \tag{1}$$

with a functional $F : X \to \mathbb{R}^n$, where X is the Banach space $X = C([-h, 0], \mathbb{R}^n)$ of continuous functions $\phi : [-h, 0] \to \mathbb{R}^n$ with fixed $h > 0$, and norm given by $|\phi| = \max_{-h \leq s \leq 0} |\phi(s)|$. A simple-looking example for (1) is

$$\dot{x}(t) = -ax(t) + f(x(t-1)) \tag{2}$$

where $n = 1$, $h = 1$, $F(\phi) = -a\phi(0) + f(\phi(-1))$ with $a \geq 0$ and a smooth $f : \mathbb{R} \to \mathbb{R}$.

Under mild conditions on F, the solutions generate a continuous semiflow Φ. The global attractor $\mathcal{A}$ of the semiflow Φ is the main object to study, whenever it exists, as it contains most of the relevant qualitative information on the long time ($t \to \infty$) behavior of the solutions. In our earlier works we also contributed to the understanding of the dynamics generated by DDEs, in particular equation (2). There are several deep results for the monotone feedback case, that is, for the case when f is monotone in equation (2).

The aim of this project was the following:

- Study equations with nonmonotone feedback functions, in particular, for the famous Mackey-Glass type equations in order to understand more about the dynamics.
- Describe additional possible structures of the global attractor for equation (2), and prove bifurcation results.
- Consider neural field models with transmission delays which do not fit into the classical framework of DDEs. Develop new analytical and numerical methods.
- Study equations with state-dependent delays motivated by real world applications.
- Applications.

2. The main results

2.1. Mackey-Glass type equations. Mackey and Glass introduced in 1977 the equation

$$y'(t) = -ay(t) + bg_k\Big(y(t-1)\Big) \tag{3}$$

to model physiological control systems. Here $b > a > 0$, $m > 0, k > 0$ are parameters, and

$$g_k(\xi) = \frac{\xi^m}{1 + \xi^k}.$$

This single equation motivated a wide range of research in DDEs. Several numerical results suggest that the dynamics of (3) can be complex. However, rigorous mathematical proofs were not available until our results.

We introduced the limiting version

$$x'(t) = -ax(t) + bg(x(t-1)) \tag{4}$$

of (3), where

$$g(\xi) = \lim_{k\to\infty} g_k(\xi) = \begin{cases} \xi^m & \text{for } 0 \le \xi < 1 \\ \frac{1}{2} & \text{for } \xi = 1 \\ 0 & \text{for } \xi > 1 \end{cases}$$

In the paper [7] we proved that for some parameter values $b > a > 0$, $m = 1$, and $k > 0$ sufficiently large, there exists a periodic orbit. The orbit is obtained by using a fixed point theorem applied to a certain return map. To prove this, we used a hypothesis (H), which assumed that the parameters $b > a > 0$ are such that there exists a periodic solution to the limiting Mackey-Glass equation (4). The periodic orbit of the Mackey-Glass equation is hyperbolic, orbitally stable, exponentially attractive with asymptotic phase and close to the orbit of the limiting Mackey-Glass equation We gave analytical tools to guarantee hypothesis (H), and developed a rigorous computer-assisted numerical technique to verify (H). Some of the obtained stable periodic solutions had complicated looking structures.

In the paper [15] we gave a nontrivial extension of the results of [7] to the case when the parameter $m > 0$ is arbitrary. The case $m = 1$ is the famous Mackey-Glass equation, the case $m > 1$ appears in population models with Allee effect, and the case $m \in (0, 1)$ arises in some economic growth models. The obtained stable periodic orbits may have complicated structures.

The paper [11] contructs stable periodic orbits with large minimal periods for equation (3) when $m = 1$ and k is large.

The manuscripts

(i) G. Benedek, T. Krisztin: Connecting orbits for delay differential equations with unimodal feedback
(ii) T. Krisztin, Pham Le Bach Ngoc, M. Polner, R. Szczelina: Homoclinic solutions for some Mackey-Glass type equations
(iii) T. Krisztin, Pham Le Bach Ngoc: Homoclinic bifurcation for a Mackey-Glass type equation
(iv) T. Krisztin, Hoang Thi Thuy: Connecting orbits for a nonsmooth Mackey-Glass type equation

are under preparation within this project. Paper (i) establishes several types of connecting orbits between, equilibrium points and periodic orbits, and between periodic orbits. Paper (ii) constructs a homoclinic orbit for equation (3) with $m = 2$ and large k. For some parameter values, the Shilnikov type spectral conditions holds, and the result gives a good start to show rigorously chaotic behavior. Paper (iii) shows that, at the homoclinic orbit, a homoclinic bifurcations takes place, that is, for close parameter values stable periodic orbits with large minimal periods appear. The manuscript (iv) shows connecting orbits where linearization is not possible due to the lack of smoothness.

2.2. Bifurcation results for DDEs. Paper [5] considered equation (2) with a nondecreasing feedback function f_K depending on a parameter K, and verified that a saddle-node bifurcation of periodic orbits took place as K varied . The nonlinearity f_K was chosen so that it had two unstable fixed points (hence the dynamical system had two unstable equilibria), and these fixed points remained bounded away from each other as K changed. The generated periodic orbits were of large amplitude in the sense that they oscillated about both unstable fixed points of f_K.

The papers [3,6,8] gave periodic orbits by Hopf bifurcations and by direct constructions.

2.3. Delay differential equations with transmission delays. The paper [13] considered delayed neural field models as a dynamical system in an appropriate functional analytic setting. On two dimensional rectangular space domains, and for a special class of connectivity and delay functions, the authors described the spectral properties of the linearized equation. They transformed the characteristic integral equation for the delay differential equation (DDE) into a linear partial differential equation (PDE) with boundary conditions. They demonstrated that finding eigenvalues and eigenvectors of the DDE is equivalent with obtaining nontrivial solutions of this boundary value problem (BVP). When the connectivity kernel consisted of a single exponential, they constructed a basis of the solutions of this BVP that formed a complete set in L^2. This gave a complete characterization of the spectrum and wass used to construct a solution to the resolvent problem. As an application they gave an example of a Hopf bifurcation and computed the first Lyapunov coefficient.

The paper [17] studies a neural field with excitatory and inhibitory neurons, transmission delays and gap junctions. The authors investigate how the gap junctions, modeled by a diffusion term, influence the behavior of the neural field. More precisely, the authors compute the normal form coefficients up to third order so as to investigate the periodic orbits generated by Hopf bifurcations in the presence of spherical symmetry.

2.4. Equations with state-dependent delay. The paper [2] concerns a scalar differential equation with a new type of state-dependent delay modeling queueing processes. The first goal is to prove the existence of a unique global solution of the equation, and to define a continuous semiflow on a suitable phase space. This theoretical framework is applied to a class of systems modelling processes with queueing delays. These systems are used to model, e.g., computer or road networks where a bottleneck leads to delays in the processing and response of the system. The second main goal of the paper is to consider a subfamily with queueing state-dependent delays, for which, in the case of an unstable equilibrium, and to prove the existence of slowly oscillating periodic solutions. The main tool to find such periodic solutions is Browder's nonejective fixed point theorem, after a careful construction of a suitable set where Browder's theorem can be applied. A rate control model introduced by P. Ranjan and co-authors is included as a particular case, and some relevant open problems in their papers are answered here.

The paper [9] analyzes a system of differential equations with state-dependent delay (SD-DDE) from cell biology, in which the delay is implicitly defined as the time when the solution of an ordinary differential equation, parametrized by the SD-DDE state, meets a threshold. The first main result is the proof that the cell SD-DDE is well-posed and the solutions define a continuous semiflow on a state-space of Lipschitz functions. The second main result is the establishment of a compact and convex set that is invariant for finite times under the v-component of the time-t-map.

The paper [14] studies the geometric structure of the solution manifold for a general class of differential equations with state-dependent delays.

The paper [18] considers scalar state-dependent delay differential equations where the nonlinearity fulfills a negative feedback condition in the delayed term. Under suitable conditions a Morse decomposition of the global attractor is constructed, giving some insight into the global dynamics. The Morse sets in the decomposition are closely related to the level sets of an integer-valued Lyapunov function that counts the number of sign changes along solutions on intervals of length of the delay. This generalizes former results for constant delay. Two major types of state-dependent delays are given to show the applicability of the result.

2.5. Applications. The papers [4,12,16] studied different delay differential equation models appearing in epidemiology.

2.6. A discrete model. The paper [10] proves global stability of the fixed point for the delayed logistic equation $x_{n+1} = ax_n(1-x_{n-2})$ for the paramater values $1 < a \leq (1+\sqrt{5})/2$. A new approach is applied by using center manifold theory and normal form theory with rigorous computer-assited numerics.

2.7. A price model. The paper [1] proved the global stability conjecture for a price model introduced by Brunovsky, Erdélyi and Walther. The manuscript [22] introduced a second delay into the model equation, and proved bifurcation of periodic orbits from infinity.

3. Participants.

The members of the research team were Tibor Krisztin, Mónika Polner, Gabriella Vas as senior reserachers. It was a great loss when Gabriella Vas passed away in 2021. This is why the proposed research on generic properties of DDEs was not completed. Two of the young reserachers, István Balázs and János Dudás defended their PhD theses during the project. Each of the PhD students Gábor István Benedek, Pham Le Bach Ngoc and Hoang Thi Thuy wrote 2 papers which are under preparation for submission. The Phd student Anett Vörös-Kiss stopped her PhD studies.

4. Summary.

In our theoretical investigations, we developed new methods for important classes of delay differential equations. The main results:

(i) We studied the famous Mackey-Glass type equations by introducing a new approach. Stable periodic orbits with complicated structures were obtained. The existence of connecting and in particular, homoclinic orbits were shown.
(ii) Delayed neural field models were investigated as a dynamical system in an appropriate functional analytic setting.
(iii) Differental equation with different types of state-dependent delays were investigated.
(iv) We considered differential equation models for epidemic spread.

We published 18 research papers. According to the Scimago journal ranking, 17 appeared in Q1-ranked journals, and 6 in D1-ranked journals. Additional 5 papers are under preparation with the support of this project.

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Papers under preparation:

[19] G. Benedek, T. Krisztin: Connecting orbits for delay differential equations with unimodal feedback

[20] T. Krisztin, Pham Le Bach Ngoc, M. Polner, R. Szczelina: Homoclinic solutions for some Mackey-Glass type equations

[21] T. Krisztin, Pham Le Bach Ngoc: Homoclinic bifurcation for a Mackey-Glass type equation

[22] T. Krisztin, Hoang Thi Thuy: Stability and periodic solutions of a price model with delays

[23] T. Krisztin, Hoang Thi Thuy: Connecting orbits for a nonsmooth Mackey-Glass type equation