Report on the the RESEARCH 125569

Weak and strong dependence in Probability Theory, Analysis and Combinatorics

from 2017-09-01 to 2024-03-31.

István Berkes

Diophantine approximation and random walks

Let $X_1, X_2, \ldots$ be independent and identically distributed, integer valued random variables, let $S_n = X_1 + \ldots + X_n$ and let α be an irrational number. Then $\{S_n\alpha\}$, where $\{\cdot\}$ denotes fractional part, is a random walk on the circle, and in papers [2], [7], [10], [18], [29] we investigate the asymptotic properties of this random walk. The degenerate case $X_1 = X_2 = \ldots = 1$ is classical: by a well known theorem of number theory, $\{n\alpha\}$ is uniformly distributed (mod 1), i.e. the empirical distribution function of the first n terms of the sequence converges in the sup metric to the uniform distribution on $(0, 1)$. The discrepancy D_n, expressing the speed of convergence, is closely connected with the Diophantine rank γ of α, measuring how closely α can be approximated by rational numbers. The case of general X_j is still unexplored and rather complicated, and we prove several results here.

In [10] we prove that if X_1 has heavy tails $P(X_1 > t) \sim ct^{-\beta}$, $0 < \beta < 2$, then, up to logarithmic factors, the order of magnitude of D_n is $O(n^{-\tau})$ where $\tau = 1/2$ for $1 \leq \gamma \leq 2/\beta$ and $\tau = 1/(\beta\gamma)$ for $\gamma > 2/\beta$. Thus at the critical value $\gamma = 2/\beta$ the function $\tau = \tau(\beta)$ changes from constant to hyperbolic decrease. This phenomenon, caused by a combination of number theoretic and probabilistic effects, in unknown in the classical case of $\{n\alpha\}$.

In [2] we determine the exact order of magnitude of $\sup_{0\leq x\leq 1} |P(\{S_n\alpha\} < x) - x|$, thereby proving an analogue of the classical Berry-Esseen theorem for Diophantine approximation theory.

In paper [29] we prove that the 'cutoff' phenomenon for card mixing and other Markov chains, i.e. the sudden change of the convergence speed to the limiting uniform distribution, has an analogue for the random walk $\{S_n\alpha\}$ on the circle. Specifically, let $\alpha = p/q$ be a rational number badly approximable with rational numbers with denominator $< q/2$ in the sense of Diophantine approximation and let $X_1, X_2, \ldots$ be i.i.d. random variables with values jp/q, $j = 0, 1, ..., q-1$. Then the distribution of $\{S_n\alpha\}$ converges weakly to the discrete uniform distribution on the numbers $p/q, 2p/q, \ldots, (q-1)p/q$, but the speed of convergence changes at $n = q^2$: for $n \leq q^2$ the speed is polynomial and for $n > q^2$ it becomes exponential.

Finally, in [7] we investigate the case of absolutely continuous X_1 and show that in this case the sequence $\{S_n\alpha\}$ is uniformly distributed (mod 1) for all $\alpha \neq 0$ and we determine the weak limit of its centered and normed empirical distribution function in the Skorohod space $D[0, 1]$. This yields, in particular, the limit distribution of $n^{1/2}D_n$ and the analogous result for $D_n^{(p)}$, $p \geq 1$, the L^p discrepancy of the process. We also show the corresponding law of the iterated logarithm, identifying the limsup in terms of an associated reproducing kernel Hilbert space.

Almost sure central limit theory

This theory deals with the a.s. limiting behavior of logarithmic averages $T_n = (\log N)^{-1} \sum_{k=1}^{N} k^{-1} f((S_k - a_k)/b_k)$ of partial sums $S_n = \sum_{j=1}^{n} X_j$ of (possibly dependent) random variables X_j. It started with the basic papers of Brosamler (1988) and Schatte (1988) and by now we have a wide and beautiful theory of such limit theorems. Papers [26] and [33] contain new contributions to this theory. In [33] we give optimal convergence conditions in the case when the X_j are i.i.d. with finite variance and f grows with speed $\exp(cx^\gamma)$, $0 < \gamma < 1/2$. In the case of independent and bounded X_j we show that $T_n \to \int_{\mathbb{R}} f(x) d\Phi(x)$ a.s. provided that the X_j satisfy the classical Kolmogorov condition of the LIL and this result is best possible. This shows a new and surprising connection between the ASCLT and LIL. We also give an optimal condition for the a.e. convergence of T_n in terms of the remainder term of the strong Wiener approximation of the partial sums S_n, revealing another interesting feature of the ASCLT.

Paper [26] gives, generalizing several earlier results in the field, optimal conditions for a sequence of random variables to satisfy the almost sure central limit theorem along a given sequence of integers.

Lacunary series

Our main result in this field is the lecture note [32] which yields a survey of the field of lacunary series in the past 50 years. The fact that sufficiently thin subsequences of general function series behave like independent random variables is a basic observation of analysis and probability theory and it played an important role in the early years of modern probability theory. The first survey of this field was published by Gaposhkin in 1966 (Lacunary series and independent functions, Uspekhi Mat. Nauk 21(6) (1966), 3–82) and it remains an important reference work until today. But in the past 50 years considerable progress took place in the field, with several new results and lines of research emerging, among others the subsequence principle, critical phenomena, connections with number theory and Diophantine approximation, applications in Banach space theory, etc. Thus publishing a new overlook of the field became important and the work [32], under print in the Springer Lecture Note series, aims to fulfill this task.

Papers [1] and [9] contain new results in this field. In [1] we investigate the asymptotic properties of lacunary series with random frequencies, specifically we determine the limit distribution of $N^{-1/2} \sum_{k=1}^{N} \sin n_k x$, where n_k are independent random variables, uniformly distributed over disjoint intervals $I_1, I_2, \ldots$. The limit distribution is mixed Gaussian, depending sensitively on the length of the intervals I_k and the distances between the consecutive I_k's. In [9] we prove the law of the iterated logarithm for exponential sums $\sum_{k=1}^{N} \exp(2\pi i n_k x)$, extending the Salem-Zygmund law of the iterated logarithm for random n_k with independent gaps $n_{k+1} - n_k$.

Trimming and parameter estimation

Paper [25] deals with the trimming of autoregressive processes. Trimming is a standard method to decrease the effect of large sample elements in statistical procedures. For example, by a result of Csörgő, Horváth and Mason, partial sums of moderately trimmed i.i.d. random variables in the domain of attraction of a stable law are asymptotically normal. In [25] we investigate the asymptotic behavior of the trimmed least square estimator of AR(1) processes with stable errors.

St. Petersburg paradox

The St. Petersburg paradox (Nicolaus Bernoulli 1713) is one of the oldest problems of probability theory, investigated in countless papers in the past 300 years. It also provided the motivation for the utility theory of von Neumann and Morgenstern. A breakthrough result in the field is due to Martin-Löf (1985) who showed that, letting S_n denote the accumulated gain of the player in n games, $(S_n - n\log_2 n)/n$ has semistable limit distributions along exponential subsequences $n_k = [c^k]$, $1 < c \le 2$. This result was extended in various directions by S. Csörgő and coauthors. In paper [8] we prove that the process $\{S_n - n\log_2 n,\ n \ge 1\}$ can be approximated by a semistable process $\{L(n), n \ge 1\}$ with error term $O(\sqrt{n}\,(\log n)^{1+\varepsilon})$ a.s. and surprisingly, the error term is asymptotically normal, exhibiting an unexpected central limit theorem in St. Petersburg theory.

Péter Major

In paper [22] Dyson's classical r-component hierarchical model is investigated with a Hamiltonian function which has a continuous $O(r)$-symmetry, $r \ge 2$. This is a one-dimensional ferromagnetic model with a long range interaction potential $U(i,j) = -l(d(i,j))d^{-2}(i,j)$, where $d(i,j)$ denotes the hierarchical distance. We are interested in the case when $l_n = l(2^n)$, $n = 1, 2, \ldots$, is an increasing sequence, with a sub-exponential growth as $n \to \infty$. For a class of free measures, we prove a conjecture of Dyson. This conjecture states that the convergence of the series $l_1^{-1} + l_2^{-1} + \ldots$ is a necessary and sufficient condition for the existence of phase transition in the model under consideration, and the spontaneous magnetization vanishes at the critical point, i.e., there is no Thouless' effect. We find, however, that the distribution of the normalized mean spin at the critical temperature T_c tends to the uniform distribution on the unit sphere in $\mathbb{R}^r$ as the volume tends to infinity, a phenomenon which resembles the Thouless effect. We prove that the limit distribution of the normalized mean spin is Gaussian for $T > T_c$, and it is non-Gaussian for $T \le T_c$. We also show that the density of the limit distribution of the normalized mean spin for $T \le T_c$ is a nice analytic function which can be found from the unique solution of a nonlinear fixed point integral equation. Finally, we determine some critical asymptotics and show that the divergence of the correlation length and magnetic susceptibility is super-polynomial as $T \to T_c$.

Another subject of Péter Major's investigation non-Gaussian is limit theorems for non-linear functionals of vector valued stationary Gaussian random fields. This is a continuation of his joint work with R. L. Dobrushin in the scalar valued case which was described in detail in his Springer Lecture Notes in Mathematics 849, "Multiple Wiener–Itô Integrals with Applications to Limit Theorems".

The motivation for this study was a result in the paper "Limit theorems for non-linear functionals of a stationary Gaussian sequence of vectors" (1994) by Arcones. One goal of this paper was to prove the above mentioned multivariate generalization of my result with Dobrushin. But the result in this work (even its formulation) was erroneous. It turned out that the right proof demands a non-trivial generalization of the Lecture Note 849. This is done in the papers [16] and [31].

The proof of the above mentioned limit theorems are based on some ideas of Itô and Dobrushin. Itô worked out a useful method to represent the non-linear functionals of Gaussian random fields by means of multiple stochastic integrals. In the case of stationary fields Dobrushin worked out such a version of this method where he could unify Itô's method with the application of Fourier analysis. This turned out to be a very useful approach. But it worked in its original form only for scalar valued processes. The goal of the papers [16] and [31] was to give such a generalization of this theory which is applicable also in the vector valued case. To carry out this program several technical problems had to be overcome.

In paper [16] the existence of the spectral measure of a vector-valued stationary Gaussian random field is proved and the vector-valued random spectral measure corresponding to this spectral measure is constructed. The most important properties of this random spectral measure are formulated, and they enable us to define multiple Wiener–Itô integrals with respect to it. This theory is worked out also for so-called generalized random fields and generalized random spectral measures, because they are needed in the formulation of the limit theorems we are interested in. Then an important identity about the products of multiple Wiener–Itô integrals, called the diagram formula is proved. An important consequence of this result, the multivariate version of Itô's formula is presented. It shows a relation between multiple Wiener–Itô integrals with respect to vector-valued random spectral measures and Wick polynomials. Wick polynomials are the multivariate versions of Hermite polynomials. In [31] we show With the help of Itô's formula that the shift transforms of a random variable given in the form of a multiple Wiener–Itô integral can be written in a useful form. This representation of the shift transforms makes possible to rewrite certain non-linear functionals of a vector-valued stationary Gaussian random field in such a form which suggests a limiting procedure that leads to new limit theorems. Finally, a result of [31] shows when this limiting procedure can be carried out, i.e., when the limit theorems suggested by our representation of the investigated non-linear functionals are valid.

Tamás F. Móri

Stochastic processes and measures of dependence with importance in statistics.

These are theoretical papers from the boundary between dependence and independence, with direct practical benefit in statistics and stochastic modelling: [5], [17], [23], [28]. Paper [21] is a bit out of line, as it is a tribute article on the occasion of Rényi's hundredth birthday, but it also discusses recent developments in connection with quantitative and qualitative independence.

Recently new methods for measuring and testing dependence have appeared in the literature. One way to evaluate and compare them with each other and with classical ones is to consider what are reasonable and natural axioms that should hold for any measure of dependence. In [5] we propose four natural axioms for dependence measures and establish which axioms hold or fail to hold for several widely applied methods. All of the proposed axioms are satisfied by distance correlation. We prove that if a dependence measure is defined for all bounded non constant real valued random variables and is invariant with respect to all one-to-one measurable transformations of the real line, then the dependence measure cannot be weakly continuous. The lack of weak continuity means that as the sample size increases the empirical values of a dependence measure do not necessarily converge to the population value.

These axioms can be used for dependence measures of random variables that take values in Hilbert spaces. However, existing measures either do not work in general (not Hilbert) spaces or they do not satisfy those axioms. The earth mover's correlation introduced in [17] applies in general metric spaces and satisfies the four axioms (two of them in a weaker form).

In [19] it is shown that for any fixed Pearson correlation coefficient strictly between 0 and 1, the distance correlation coefficient can take any value in the open unit interval (0,1). This result has already been cited by more than 200 papers reporting applied researches in several distinct areas.

The energy test of normality is an affine invariant consistent test based on energy distance. Parameters of the limit distribution are eigenvalues of an integral operator. In [23] explicit integral equations are derived for the eigenvalue problem for simple and composite hypotheses of normality, and they are solved by a variation of Nyström's method for four cases of known or estimated parameters.

Paper [28] deals with pseudorandom processes that appear to be random but they just imitate certain properties of random/stochastic processes. Our starting point is the random walk approximation of Brownian motion, that is, in terms of the sums of iid Bernoulli random variables. Here we replace them by the most classical orthogonal functions like sine and cosine. It turns out that their partial sums have limit distributions without dividing them by the usual square root of n. More general functions lead to unexpected new challenges. Our summands are always identically distributed but not independent, not even exchangeable. The price we need to pay is that limit distributions of pseudo-random series might depend on the order of terms.

Behavior of evolving random graphs and its applications.

We continued the analysis of a recently introduced increasing graph process. This model is motivated by pairwise collaborations, and it is driven by time-dependent branching dynamics. In [6] we consider a slightly generalized version of the original model, and compare the stable age distributions of the edges for physical, and for biological age. The former is simply the time elapsed from birth, while the latter is measured by the number of offspring, and the death rate is connected with it. Somewhat surprisingly, we find that the two distributions have completely different tail behaviour, although the expected value of their biological age is a linear function of the physical age. Paper [12] also deals with randomly evolving graph processes. Due to their popularity, there exist randomized versions of many recursively defined graph models. One of them is the cherry tree model introduced to improve Bonferroni type upper bounds on the probability of the union of random events. Here we consider a substantially extended random analogue of it, embedding it into a general time-dependent branching process. We establish several properties of the model, such as the probability of extinction, the asymptotic number of vertices or edges, the evolution of the degree of a fixed vertex, etc.

Paper [13] is devoted to the study of some properties of supercritical Crump–Mode–Jagers (CMJ) processes. More precisely, we are interested in conditions sufficient for a suitably scaled CMJ process (counted by appropriate random characteristics) to have finite k-th moments, uniformly in time. In this paper we present such conditions. This research is motivated by the problem of describing the asymptotic behaviour of the maximal degree in a recently introduced random graph process. Our results are strong enough to attack the order of magnitude of the maximal degree.

Most randomly growing graph models, like the popular scale free networks, lead to sparse graphs, but a large number of papers have been devoted to dense graphs, too. Less attention has been paid to the interesting class of models with moderate density, i.e., where the number of edges is asymptotically a power function of the number of vertices, with exponent strictly between 1 and 2. In [24] we introduce such a model. We analyze its asymptotic properties in terms of the number of edges, namely, the (random) number of vertices, the degree of a fixed vertex, the proportion of vertices with degree greater than a fixed multiple of the average degree, and the maximal degree.

Application of stochastic models to biological systems.

There are two groups of papers concerning this topic. The first group consists of papers that are basically of biological nature, their research problems come from biology, and their results have sense and importance in biology, but the mathematical models and the methods of analysis are stochastic. The exact descriptions of the models, together with the mathematical details of the analysis are often hidden in Appendices or in files of Supplementary Information; nevertheless they have a crucial role in the researches in question. Such papers are [4], [11], [15], [20], [30], [34], [35], [36]. The areas involved in the research include parent-offspring communication, evolutionary roots of human moral behavior, pollineation and fertilization in greenhouses, origin of eukaryotes and organellogenesis, kleptoparasitism, filial cannibalism.

In the second group there are strictly mathematical papers with consequences and applications in biology, mostly in evolutionary game theory. Recently, matrix games under time constraints have been investigated, and the (monomorphic) evolutionarily stable strategies (ESS) have been characterized. In [3] we are interested in how the ESS is related to the existence and stability of equilibria for polymorphic populations. We point out that, although the ESS may no longer be a polymorphic equilibrium, there is a connection between them. In the case when there are only two pure strategies, a polymorphic equilibrium is locally asymptotically stable under the replicator equation for the pure-strategy polymorphic model if and only if it corresponds to an ESS. In [14] we give two further static (monomorphic) characterizations which are analogues of those known for classical evolutionary matrix games.

In [27] we deal with some matrix games that are interesting in biology. We show that if the interaction between two players is repetitive, there are conditions on the payoff matrix guaranteeing that the best reply player can replace the player following a mixed evolutionarily stable strategy (Nash equilibrium). This can even occur when the mean number of repetitions is arbitrarily small. Thus, in the presence of repetitions, the mixed evolutionarily stable strategy is not necessarily evolutionarily stable in the Darwinian sense.