Final Report for the Grant of National Research, Development and Innovation Office NKFIH reg.no. PD 124875

In the Research plan of the Grant of National Research, Development and Innovation Office NKFIH reg.no. PD 124875 we assigned 3 different topics of mathematical analysis, which are the following:

1) Generalization of Gauss-composition of means for the case of matrix means;

2) Examination of different kinds of preserver problems;

3) Studying of generalized Wright convex functions.

In the confines of the last three years we mainly considered questions that appear in the above-mentioned areas.

Firstly, we have to mention that the original problem of the topic 1) is not solved. However, we would like to emphasize that we obtained results on the area of matrix means and means of real numbers, too. We start with a result on matrix means of positive definite matrices, which can be connected to the topics 1) and 2). We have defined generalized quasi-arithmetic means of positive definite matrices, jointly with Gergő Nagy. Moreover, under certain quite general conditions we have described the transformations on the sets of all invertible elements of the algebra of all bounded linear operators on a Hilbert space leaving a norm of a quasi-arithmetic mean invariant. The motivation of that result comes from the paper [14], where the authors concerned with the quasi-arithmetic means of positive definite matrices. In that paper, the authors investigated transformations on Hermitian matrices with invariance properties related to quasi-arithmetic means. These means are defined via the well-known formula used in the case of real numbers.

We also obtained a new result with respect to generalized quasi-arithmetic means. We investigated monotonicity properties of that means. We pointed out that there are several other problems which are frequently studied concerning means of real numbers, e.g. the problem of homogeneity and equality. In those problems, certain conditions hold for all values of the variables of the means under consideration. We could study their counterparts for the generalized quasi-arithmetic operator mean and apply those conditions to scalar operators in the set of all self-adjoint operators. In this way, we inferred that they hold also for generalized quasi-arithmetic means as a scalar mean. Therefore, the investigation of the mentioned problems in the case of the operator mean can be reduced to their study in the setting of generalized weighted quasi-arithmetic means of real numbers. However, this is not the case with monotonicity. We proved the interesting property that in a quite general setting, generalized weighted quasi-arithmetic means on self-adjoint operators are not monotone in their variables. Moreover, we investigated the relation of the generalized quasi-arithmetic means with the Kubo-Ando means and we obtained that the common members of their class and the set of the latter means are weighted arithmetic means.

The paper [24], containing the above-mentioned results, is published at the Electronic Journal of Linear Algebra.

In the third year of the grant, we considered quasi-arithmetic means of real numbers. In a joint manuscript with Pál Burai and Gergely Kiss we studied a well-known result of János Aczél (see [1]), where he established a characterization of 2-variable quasi-arithmetic means in 1948. By his result, it is known that a two-variable real function on [a,b]x[a,b] is a quasi-arithmetic mean if and only if it is continuous, reflexive, partially monotone increasing, symmetric, bisymmetric. We emphasize that this more than 60 years old result is one of the most fundamental theorems of the area of means. There

are many results which can be considered as a generalization of the mentioned result. It is a natural question whether all the mentioned assumptions are needed to characterize a quasi-arithmetic mean or are there are there any redundant assumptions in the original theorem of Aczél. Our aim was to check the regularity property. Fortunately, we were able to show that continuity property can be omitted, i.e. the theorem of Aczél remains true without continuity. We obtained a tricky proof for that important generalization. Furthermore, concerning the same result of János Aczél, there is an other direction of the generalization. Namely: is it possible to characterize any other class of means if we eliminate one of the original assumptions? It is known, that if a two-variable real function on [a,b]x[a,b] is continuous, reflexive, partially monotone increasing, bisymmetric, but not necessarily symmetric, than it is a weighted quasi-arithmetic mean (see [2]). Hence, one can ask whether the mentioned result is true in the noncontinuous case or not, i.e. the continuity property is redundant in the characterization of weighted quasi-arithmetic means or not. Since the last-mentioned problem seems to be solved, as well, we plan to submit a manuscript containing the generalizations of the characterization theorems of quasi-arithmetic and weighted quasi-arithmetic means. Moreover, we have to note, that the multivariable case of quasi-arithmetic means (not only two-, but more variable means) was studied by many mathematicians (see [3], [4], [20] and the references cited therein) and it has many applications, as well. Hence, our plan is to investigate our noncontinuous generalization of Aczél's result in the case of multivariable quasi-arithmetic means.

Concerning the previous questions, we emphasize, that we are going to study Aczél's theorem in the context of matrix/operator means, as well. However, for that kind of investigation quite new ideas are needed.

Finally, we mention that we have won a new Grant (Wacław Felczak Foundation) with Hungarian (Pál Burai, Zsolt Páles) and Polish (Justyna Jarczyk, Witold Jarczyk and Paweł Pasteczka) mathematicians, where we are going to plan to study means of real numbers and matrix means, as well.

Now, we turn to the topic 2) Examination of different kinds of preserver problems. Besides, the mentioned jointly result with Gergő Nagy we considered an other topic. Firstly, we have explored the structure of certain surjective generalized isometries (which are transformations that leave any given member of a large class of generalized distance measures invariant) of the set of positive invertible elements in a finite von Neumann factor with unit Fuglede-Kadison determinant.

This research topic was motivated by the following results: In [21] Moakher studied in details the manifold of n by n positive definite (PD) matrices with unit determinant. Moreover, in the paper [12], the authors investigated the same structure because of its interesting connections to the space of so-called diffusion tensors. In fact, they also studied the set of all PD matrices with determinant c>0, which is a so-called totally geodesic submanifold of the manifold of positive definite matrices. Motivated by these facts, in [23], among others, the problem of establishing the complete description of „generalized isometries" with respect to generalized distance measures (of a special form) was solved on the set of positive definite matrices with determinant 1. The generalized distance measure is induced by a real function and a norm defined on the set of positive definite matrices that satisfy some mild assumptions. On the other hand, the generalized isometries between the positive cones of von Neumann factors were studied and described by Lajos Molnár in [22]. Finally, we recall that there are several notions of the determinant of operators on infinite dimensional Hilbert spaces, as well. The most natural one is the Fredholm determinant. The Fredholm determinant has the disadvantage that it is defined only for quite small subsets of operator algebras. Another definition of the determinant, which was developed by Fuglede and Kadison [13] does not have this disadvantage.

In our first result with Marcell Gaál and Gergő Nagy we have described the structure of all transformations of the set of positive invertible elements of a finite Neumann factors with unit Fuglede-Kadison determinant.

We conclude that any such map originates from either an algebra *-isomorphism or an algebra *-antiisomorphism of the underlying operator algebra.

Finally, we emphasize that just as in the paper [23], besides the set of positive invertible elements with unit Fuglede-Kadison determinant, we have also considered, for a given number c>0, the collection of all positive invertible elements of a finite von Neumann algebra with Fuglede-Kadison determinant equal to c. And we have presented the corresponding structural theorem of „generalized isometries". The above-mentioned results can be found in the paper [15], which is appeared in Taiwanese Journal of Mathematics.

Concerning the topic 3) (Studying of generalized Wright convex functions) we investigated two problems which were motivated by the paper [9]. In that paper, the authors considered the so-called two-dimensional Schur-convex and t-Schur convex functions. They have introduced an auxiliary function. Using the global minima and the monotonicity of that function they have given a characterization of 1/2-Schur-convex functions and symmetric Schur-convex functions, respectively. Our aim was to extend the previous results to the case of multivariable functions, where the notion of Schur-convexity is induced by special doubly stochastic matrices. In fact, we investigated symmetric T-Schur convex functions, where T is a circulant, doubly stochastic matrix generated by a probability distribution vector. In a particular case we gave a complete characterization of that kind of functions. Namely, using the global minima we could characterize symmetric Schur-convex functions if the Schur-convexity is induced by a special matrix. The result can be found in the paper [10].

Fortunately, later on we were able to give a characterization with the help of „monotonicity", as well. Hence, we reached a real generalization of the result appearing in [9]. Moreover, we examined circulant, doubly stochastic matrices and we obtained some structural results on that set of matrices. The mentioned results are summarized in a joint manuscript with Pál Burai. We plan to submit that manuscript after some further examination.

As we mentioned, motivated by the same paper [9], we obtained an other result, which is related to the topic "Studying of generalized Wright convex functions", as well. In a further theorem of the paper [9] the connection between a Hermite-Hadamard type inequality and a Jensen-type inequality was examined in the case of two-variable functions. To prove that result, the authors applied a Korovkin-type result, which appeared in the paper [19]. We emphasize that the theories of Hermite-Hadamard type inequalities and Korovkin-type theorems are important by itself, as well. They are widely investigated and there are many generalizations of them (see e.g. [6], [7], [8] and [16], [17], respectively).

Our aim was to extend the mentioned result to the case of multivariable functions, as well. That is, we wanted to describe the connection between a multivariable Hermite-Hadamard type inequality and a Jensen-type inequality. More precisely, we investigated symmetric, continuous functions whose domain is the standard simplex of n-dimensional Euclidean space that satisfies an Hermite-Hadamard type inequality. We verified that every such function is necessarily Jensen-convex. Just as in the case of two-variables functions we had to prove a Korovkin-type result, as well, which can be interesting by itself. Besides the Korovkin-type result we had to prove further properties of circulant, doubly stochastic matrices generated by probability distribution vectors. We also obtained explicit formulas of different integrals over simplices, which was applied in the proof of the main theorem, as well. Moreover, as applications of our main theorem we obtained several theorems that can be considered as new characterizations of convex, quasi-convex and strong convex functions. The most important advantage of that kind of new characterization results is that the differentiability of the studied functions is not assumed. The above-mentioned results are summarized in a joint manuscript with Pál Burai and Judit Makó [11] and it is accepted for publication in Mathematical Inequalities and Applications.

Concerning the above-mentioned results, we have further ideas to obtain new results. For example, we are going to study functions that satisfy “multivariable Hermite-Hadamard-Fejér type” inequalities.

Finally, I would like to emphasize that in [5] we obtained a result which is not closely related to the research plan. In this result we obtained a generalization of the stability of some Banach lattice-valued functional equation) with the addition replaced in the Cauchy functional equation by lattice operations and their combinations. The concerning results were presented at „The 18th Debrecen-Katowice Winter Seminar” with the title Stability of a functional equation on Banach lattices.

We sum up, that during the of the Grant of National Research, Development and Innovation Office NKFIH reg.no. PD 124875, four papers are appeared in different journals for publication or published ([5], [10], [15], [24]), one further paper is accepted for publication in Mathematical Inequalities & Applications ([11]) and we have two further manuscripts, which are planned to be submitted for publication (on characterization of means and description of circulant, doubly stochastic matrices). Moreover, the obtained results were presented at 10 different conferences (from which 9 are international ones). We would like to emphasize two conferences with titles “Preserver weekend in Szeged” and “The 56st International Symposium on Functional Equations”. Concerning the first one, as the title shows, the aim of the workshop was to present recent results on the research area commonly referred to as "Preserver Problems". On the second conference, there was a special session on Schur-convex functions, where I also delivered a talk in that section. The speakers also confirmed that Schur-convex functions appears not only in the area of mathematics, but they play also an important role in widely linear processing, majorization theory, cellular networks (see e.g. [18] and the references cited there). This fact motivated us to continue our studying on Schur-convex functions.

Unfortunately, in the past half a year there was not a real opportunity to participate any conference, and we have more administration duties at the university, which made the research work difficult.

The final author’s version of our papers and presented talks during the grant can be found on the following website: https://arato.inf.unideb.hu/szokol.patricia/kutatas.html.

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