FINAL REPORT on the
NKFI FK 123962 Young Researcher's Grant
Large-scale behavior of random spatial processes and interacting particle systems

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In this final report, we summarize the research conducted in the period between 1 Sep 2017 and 31 Aug 2022 supported by the NKFI FK 123962 Young Researcher's Grant of the National Research, Development and Innovation Office of Hungary (NKFI). Our research activity was hosted at the Department of Stochastics at the Budapest University of Technology and Economics (BME).

Our goal for this research period was to deepen the understanding of the theory of random spatial processes and interacting particle systems and as a result, to publish our results in collaboration with our international co-authors in high-quality referred journals, moreover to give research talks about our results at international conferences and seminars. Another goal of ours was to expand the probability school of BME by involving young talent in our research and to start the supervision of PhD students. We are proud to say that we have achieved these goals. In numbers: we have submitted 13 publications, 10 of which are now published (or accepted for publication) in high quality journals (two papers appeared in journals with D1 ranking, four with Q1 ranking and three with a Q2 ranking), moreover 6 out of the 12 publications of the senior participants of the project (Ráth, Vető) are co-authored by the younger generation (Borbényi, Fekete, Kiss, Molontay, Rokob, Szőke, Talyigás, Varga), who (at the time of the writing of our papers) were either MSc of Phd students. In 2021, two students (Keliger, Szőke) started their PhD under the supervision of Balázs Ráth. During the research period of the grant NKFI FK 123962, the two senior participants gave 18 conference and seminar talks (see the end of this document for the detailed list of our talks).

The rest of this document contains a brief overview of our research results. Note that the numbering of subsections coincides with that of the research proposal, which in turn reflects the subdivision of our research into themes and topics. Let us note that mathematical research is inherently unpredictable: on the one hand, some of our best results from the research period of the NKFI FK 123962 grant did not even exist as a plan back in 2017 when we wrote the research proposal; on the other hand, we did not (fully) cover some of the topics that we mentioned in the research proposal for various reasons that we outline below.

1 Correlated percolation models [2, 21, 23]

1.1 Geometry of sparse random interlacements

Jointly with Eviatar Procaccia and Artëm Sapozhnikov, we are still planning to prove the results outlined in Section 1.1 of the research proposal. In fact, E. Procaccia and his co-authors have already made some progress in this direction, see [7]. Let us devote the rest of this subsection to the description of our results [2, 23], both of which are related to either random interlacements or finitary random interlacements.

1.1.1 Percolation of worms [23]

A worm is a simple random walk trajectory on $\mathbb{Z}^d$ with a random length L. Independently from each vertex of $\mathbb{Z}^d$ we start $\mathrm{Poi}(v)$ i.i.d. worms (i.e., simple random walk trajectories). It is easy to show that if the second moment of L is finite then for small values of v the set $\mathcal{S}(v)$ of sites visited by the worms consists of finite connected components, while for big values of v it contains an infinite connected component (i.e., the model exhibits percolation phase transition). We give a sufficient condition on the distribution of L (which is only weaker than having infinite second moment by a $\log(\log(\cdot))$ correction of the probability mass function of L) under which $\mathcal{S}(v)$ contains an infinite connected component for all $v > 0$ (i.e., there is no phase transition).

1.1.2 Random interlacement is a factor of i.i.d. [2]

The random interlacement point process (introduced in [24]) is a Poisson point process on the space of labeled doubly infinite nearest neighbour trajectories modulo time-shift on a transient graph G. We show that the random interlacement point process on any transient transitive graph G is a factor of i.i.d., i.e., it can be constructed from a family of i.i.d. random variables indexed by vertices of the graph via an equivariant measurable map (i.e., a map T which intertwines the natural action of the group of isomorphisms of G on the domain and the range of T). Our proof uses a variant of the soft local time method (introduced in [16]) to construct the interlacement point process as the almost sure limit of a sequence of finite-length variants of the interlacement model with increasing length. We also discuss a more direct method of proving that the interlacement point process is a factor of i.i.d. which works if and only if the graph G is non-unimodular.

1.2 Voter model percolation [21]

Jointly with Daniel Valesin, we are still planning to prove the results outlined in Section 1.2 of the research proposal pertaining to percolation of the stationary distribution of a classical interacting particle system, the voter model. Let us devote the rest of this subsection to the description of our results [21] about the percolation of the stationary distribution of another classical interacting particle system: in the R-spread out contact process, each site of $\mathbb{Z}^d$ can be healthy or infected. An infected site heals at rate one and at rate λ it infects a uniformly chosen vertex within a ball of radius R. One may view a configuration sampled according to the upper stationary measure of the contact process as a correlated nearest neighbour site percolation model. We identify the limit of the percolation threshold value of λ as $R \to \infty$. Our result implies that the answer to an open question of [13] is affirmative in the R-spread-out case.

2 Dynamic random graph models [3, 15, 19, 20, 22]

Apart from the topics outlined in the research proposal, we also studied an interesting new random graph model (introduced in [10]) called color-avoiding percolation. We discuss our results pertaining to color-avoiding percolation at the end of this subsection.

2.1 Mean field models of self-organized criticality [3, 19, 20]

Jointly with Dominic Yeo, we are still working on our proof of (the subcritical statement of) Conjecture 1.1 of [18].

2.1.1 Age evolution in the mean field forest fire model via multitype branching processes [3]

We study the distribution of ages in the mean field forest fire model introduced in [17]. This model is an evolving random graph whose dynamics combine Erdős–Rényi edge-addition with a Poisson rain of lightning strikes. All edges in a connected component are deleted when any of its vertices is struck by lightning. We consider the asymptotic regime of lightning rates for which the model displays self-organized criticality. The age of a vertex increases at unit rate, but it is reset to zero at each burning time. We show that the empirical age distribution converges as a process to a deterministic solution of an autonomous measure-valued differential equation. The main technique is to observe that, conditioned on the vertex ages, the graph is an inhomogeneous random graph in the sense of Bollobás, Janson and Riordan. We then study the evolution of the ages via the multitype Galton–Watson trees that arise as the limit in law of the component of an identified vertex at any fixed time. These trees are critical from the gelation time onwards.

2.1.2 Frozen percolation on the binary tree [19, 20]

In frozen percolation, i.i.d. uniformly distributed clocks are assigned to the edges of a graph. When its clock rings, an edge opens provided neither of its endvertices is part of an infinite open cluster; in the opposite case, it freezes. Aldous showed in [1] that frozen percolation is well-defined on the infinite 3-regular tree. However, it was not known whether the set of frozen edges is a measurable function w.r.t. the sigma-algebra generated by the clocks. We give a negative answer to this question in [19]. An essential role in the proof is played by oriented frozen percolation on a novel scale invariant object, the marked binary branching tree (MBBT). The MBBT is the family tree of a rate one binary branching process, on which points have been generated according to a rate one Poisson point process, with i.i.d. uniformly distributed activation times assigned to the points. In frozen percolation on the MBBT, initially, all points are closed, but as time progresses points can become either frozen or open. Points become open at their activation times provided they have not become frozen before. Open points connect the parts of the tree below and above it and one says that a point percolates if the tree above it is infinite. We consider a version of frozen percolation on the MBBT in which at times of the form θ^n, all points that percolate are frozen. The limiting model for $\theta \to 1$, in which points freeze as soon as they percolate gives back the model studied in [19]. In [20] we extend this result by showing that there exists a $0 < \theta^* < 1$ such that the model is endogenous for $0 < \theta \leq \theta^*$ but not for $\theta^* < \theta < 1$. This means that for $\theta \leq \theta^*$, frozen percolation is a.s. determined by the MBBT but for $\theta > \theta^*$ one needs additional randomness to describe it.

2.2 Village model and quantum random graph

We are still interested in the study of the model outlined in Section 2.2 of the research proposal, but the only scientific work that attests our dedication to this topic is the BSc thesis of Vinh Hung Nguyen [8], written under the supervision of Balázs Ráth. This BSc thesis contains the results of multiple computer simulations that are in line with the conjectures formulated in Section 2.2 of the research proposal.

Color-avoiding percolation [15, 22]

On the complexity of color-avoiding site and bond percolation [15]

The mathematical analysis of robustness and error-tolerance of complex networks has been in the center of research interest. On the other hand, little work has been done when the attack-tolerance of the vertices or edges are not independent but certain classes of vertices or edges share a mutual vulnerability. In this study, we consider a graph and we assign colors to the vertices or edges, where the color-classes correspond to the shared vulnerabilities. An important problem is to find robustly connected vertex sets: nodes that remain connected to each other by paths providing any type of error (i.e. erasing any vertices or edges of the given color).

This is also known as color-avoiding percolation. In [15], we study various possible modeling approaches of shared vulnerabilities, we analyze the computational complexity of finding the robustly (color-avoiding) connected components. We find that the presented approaches differ significantly regarding their complexity.

Color-avoiding percolation in edge-colored Erdős-Rényi graphs [22]

We study a variant of the color-avoiding percolation model introduced by Krause et al., namely we investigate the color-avoiding bond percolation setup on (not necessarily properly) edge-colored Erdős–Rényi random graphs. We say that two vertices are color-avoiding connected in an edge-colored graph if after the removal of the edges of any color, they are in the same component in the remaining graph. The color-avoiding connected components of an edge-colored graph are maximal sets of vertices such that any two of them are color-avoiding connected. We consider the fraction of vertices contained in color-avoiding connected components of a given size as well as the fraction of vertices contained in the giant color-avoiding connected component. Under some mild assumptions on the color-densities, we prove that these quantities converge and the limits can be expressed in terms of probabilities associated to edge-colored branching process trees. We provide explicit formulas for the limit of the normalized size of the giant color-avoiding connected component.

3 Models in the KPZ universality class [5, 6, 25, 27]

3.1 KPZ equation and directed polymer models [25]

In [25], we consider two directed polymer models in the Kardar–Parisi–Zhang (KPZ) universality class. They are the O'Connell-Yor semi-discrete directed polymer with boundary sources and the continuum directed random polymer. In this work, we introduce and use the (m, n)-spiked initial condition for the KPZ equation and the corresponding (m, n)-spiked boundary perturbations for the continuum directed polymer. We prove that the limiting fluctuations of the free energies rescaled by the 1/3rd power of time in both polymer models converge to the Borodin–Péché type deformations of the GUE Tracy–Widom distribution.

3.2 Interacting particle systems [6, 27]

In [6], we consider the limiting distribution of KPZ growth models with random but not stationary initial conditions. The one-point distribution of the limit is given in terms of a variational problem involving a Brownian motion and an independent Airy_2 process minus a parabola. We study these two components in detail. The supremum of Brownian motion minus a parabola was already analyzed in the literature and we rely on existing results on its distribution. The other component is the Airy_2 process minus a parabola where the coefficient of the parabola is a parameter in $(0, 1]$. The distribution of the supremum was known only in the case of coefficient 1 when it is equal to the GOE Tracy–Widom distribution. In the general case, we give upper and lower bounds on the right tail for the distribution of the supremum which match up to a logarithmic factor. These bounds follow by a direct study of the Airy_2 process and they can be of independent interest. We use these bounds to deduce the right tail asymptotic of the limiting distribution function of KPZ class models. This gives a rigorous proof and extends the results obtained by Meerson and Schmidt in [14].

In [27], we consider several generalizations of the totally asymmetric simple exclusion process (TASEP) which is the most fundamental model in the KPZ universality class. We investigate the asymptotic fluctuation of particle flux in the geometric q-TASEP, the geometric

q-PushTASEP and the q-PushASEP models. All particles in the first two models jump in one direction whereas jumps in both directions are allowed in q-PushASEP. We prove that the rescaled particle position converges to the GUE Tracy–Widom distribution in the homogeneous case. If the jump rates of the first finitely many particles are perturbed in the first two models, we obtain Baik–Ben Arous–Péché and finite GUE limiting fluctuations.

3.3 Non-intersecting paths and tiling problems [5]

In [5], we consider uniform random domino tilings of the restricted Aztec diamond domain which is obtained by cutting off an upper triangular part of the Aztec diamond by a horizontal line. The limiting fluctuations for the boundary of the north polar region in the unrestricted model were described using the evolution of the top curve in the corresponding non-intersecting line ensemble. We represent the correlation kernel of the non-intersecting line ensemble in terms of a random walk in a novel way.

The line of restriction in the Aztec diamond is chosen so that it asymptotically touches the arctic circle (that is the limit shape of the north polar region in the unrestricted model) and it has a non-trivial interaction with the boundary of the north polar region on its fluctuation scale. We prove that the rescaled boundary of the north polar region in the restricted domain converges to the Airy_2 process conditioned to stay below a parabola as the size of the domain tends to infinity. We prove the convergence of continuous statistics and that of finite dimensional distributions. The limit is the hard-edge tacnode process for Brownian motion obtained for non-intersecting Brownian bridges conditioned to stay below a threshold in [4]. It is a one-parameter family of processes which depends on the tuning of the threshold position on the natural fluctuation scale.

3.4 Scaling limits of random matrix models

In a research project in collaboration with Diane Holcomb (Stockholm), Gaultier Lambert (Zürich), Elliot Paquette (Ohio) and Bálint Virág (Toronto), we tried to show a functional central limit theorem for the linear statistics of the Sine_β process which appears as the local bulk limit of β-ensembles. The idea of proof employed the counting function description of the Sine_β process introduced in [26] and it was using a martingale characterization of the linear statistic and the central limit theorem for martingales. Despite of promising partial results we could not complete the proof because the quadratic variation of the martingale that appeared could not be bounded as we expected. In the meantime the papers [12] and [11] concluded the same central limit theorems using different methods.

4 Random stirring model

We could not make progress on this notoriously difficult research topic.

Further problems not mentioned in the research plan [9]

The elephant random walk is a random walk with long memory in which the walker repeats one of the previous steps with a certain probability where the repeated step is chosen uniformly at random from all the previous steps, or a new step is sampled from a given distribution. The step distribution is a coin toss in the original model, we consider general step distributions. The probability for repeating a step is the memory parameter of the model. Depending on its value, the fluctuation scale of the position of the walker undergoes a phase transition. We consider the supercritical case where the scaling is non-standard and the scaling exponent depends on the memory parameter. In [9] we identify the previously unknown first four moments of the limiting distribution in the case of general step distribution.

Conference and seminar talks given by the senior participants

1. Sept 2017: Balázs Ráth: *A moment-generating function for Erdős-Rényi component sizes,* Randomness and Graphs : Processes and Structures, Eurandom, Eindhoven, The Netherlands
2. Apr 2018: Balázs Ráth: *The window process of slightly subcritical frozen percolation,* UK Easter Probability Meeting, University of Sheffield, UK
3. June 2018: Balázs Ráth: *On The Threshold Of Spread-out Voter Model Percolation*, (invited session talk) The 40th Conference on Stochastic Processes and their Applications, Gothenburg, Sweden
4. June 2019: Balázs Ráth: *On random graphs and forest fires,* Felix-Klein Colloquium, University of Leipzig, Germany
5. Aug 2019: Balázs Ráth: *On The Threshold Of Spread-out Voter Model Percolation*, Workshop on Complex Systems, UTIA, Prague, Czech Republic
6. Sept 2019: Balázs Ráth: *Frozen percolation on the binary tree is nonendogenous*, Workshop on Large Scale Stochastic Dynamics, Oberwolfach Research Institute for Mathematics, Germany
7. March 2020: Balázs Ráth: *On the threshold of spread-out contact process percolation*, Budapest-Vienna Probability Seminar, Rényi Institute, Budapest
8. Aug 2020: Bálint Vető: *Tilings of the Aztec diamond on restriced domains,* Bernoulli–IMS One World Symposium, online conference
9. Apr 2021: Bálint Vető: *Határeloszlások felületnövekedési folyamatokban (in Hungarian),* online seminar for students, Eugene Wigner College of Advanced Studies, Budapest
10. May 2021: Balázs Ráth: *A phase transition between endogeny and non-endogeny,* Analysis and Probability Seminar (online), Chalmers University, Gothenburg, Sweden
11. May 2021: Balázs Ráth: *Percolation of worms,* Probability Seminar (online), University of Groningen, The Netherlands
12. Jul 2021: Bálint Vető: *Upper tail decay of KPZ models with Brownian initial conditions,* online conference talk, 10th World Congress in Probability and Statistics, Seoul
13. Sep 2021: Bálint Vető: *Fluctuations in random surface growth,* online seminar talk, Virtual Humboldt Kolloquium
14. Sep 2021: Bálint Vető: *Fluctuations in random surface growth,* seminar talk, Budapest–Vienna Probability Seminar
15. Oct 2021: Balázs Ráth: *Percolation of worms,* Bangalore Probability Seminar (online), India
16. Oct 2021: Balázs Ráth: *Percolation of worms,* Bristol Probability Seminar (online), UK

17. Feb 2022: Balázs Ráth: *Percolation of worms,* Delft Probability and Statistics Seminar (online), The Netherlands

18. May 2022: Balázs Ráth: *Random interlacement is a factor of i.i.d.,* Workshop on Graphs, Groups and Stochastic Processes, Erdős Center, Rényi Institute, Budapest

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