Beszámoló a K-120706 azonosítószámú
"Információelmélet és alkalmazásai" című kutatás eredményeiről

Classical information theory:

A random access model introduced by Luo and Ephremides [6] was investigated and a related old result of Imre Csiszár was extended. In Csiszár's old paper [3] one sender could choose among several codebooks that differ in the communication rate they achieve. In the extension there are several participants and the codebooks they can choose from differ also in their codeword lengths. It is proven that a similar result to that of Csiszár (which relates the error exponent to the random coding exponent, actually shows their equality) holds in this more general setting as well. One of the new features is that a certain asynchronism is caused because the beginning and end of different length codewords should also be detected. This work started already before the present project but a significant part of it was done while the project was already going on. These results were presented at the 2017 IEEE International Symposium on Information Theory (ISIT) and its extended paper version appeared in the IEEE Transactions on Information Theory in 2018.

Exponential error bounds were derived in a sparse communication model over a discrete memoryless channel where codewords are transmitted in certain time intervals and other times the channel outputs pure noise. These results were also presented in a conference paper at the IEEE ISIT in 2017. Methods used also here (a refinement of the very useful and popular method of types that uses so-called subtypes) were further developed and a longer project was started (or rather continued) to understand the possible effects of controlled asynchronism in communication over a multiple access channel. The main and somewhat surprising phenomenon discovered is already given in the title of the final paper, namely that controlled asynchronism may outperform synchronism. This means that error exponents achievable by synchronous transmission (if possible) can be superseded via controlled asynchronism, i.e. by a deliberate shift of the codewords. The paper containing the findings in this longer project about controlled asynchronism finally appeared in the IEEE Transactions on Information Theory in 2021. The paper covers the material of three earlier conference papers by its first two authors, two of which are even earlier than the start of our project. A fourth conference paper which was presented at the International Symposium on Information Theory and its Applications (ISITA) in 2018 that is related and referred to but not covered by this longer paper analyzed successive decoding for the asynchronous multiple access channel via the subtype technique, and showed that combined with controlled asynchronism it provides an alternative to rate splitting, when synchronization is possible.

During the above investigations the authors of the above paper also discovered that a controlled asynchronous code for a multiple access channel gives rise to a one-sender trellis code (a nonlinear generalization of the convolutional code).

They have put efforts into trying to understand whether the techniques used in the above investigations can turn out to be useful also in this area. While the connection to trellis codes is already exposed in the above mentioned paper these investigations still did not get to their final point and the question in the previous sentence is not yet answered. A preliminary version of the results of these investigations is available in a conference paper by Lóránt Farkas presented online at ISITA2020. In the last half year of our project a new coding method based on this conference paper has been devised. A new model in the online modul of matlab called simulink has been created, but its performance was only on par with the best state of the art LDPC (Low Density Parity Check) codes. After further analysis a possible source of the error of the model has been detected. With this in hand, a new model and a decoding method needs to be implemented, but that is not yet done. In the optimistic case this might lead to codes with better performance than the state of the art codes, but at the moment this is not more than optimistic speculation.

Applications of information theory in probability theory and related fields:

Katalin Marton's paper entitled "Logarithmic Sobolev inequalities in discrete product spaces" deals with proving so-called logarithmic Sobolev inequalities for measures on discrete product spaces, by proving inequalities for an appropriate Wasserstein-like distance. A logarithmic Sobolev inequality is, roughly speaking, a contractivity property of relative entropy with respect to some Markov semigroup. It is much easier to prove contractivity for a distance between measures, than for relative entropy, since a distance is symmetric and satisfies the triangle inequality. The method here is used to prove logarithmic Sobolev inequalities for measures satisfying a version of Dobrushin's uniqueness condition, as well as for Gibbs measures satisfying a strong mixing condition. The work presented in this paper started before the beginning of our project but the author still worked on its revision during the project and it appeared only in 2019. Unfortunately this became her last published paper.

Tamás Kói has worked with medical doctors in the area called meta-analysis which is a branch of applied statistics and as such he used his information theoretic knowledge in the analysis of data that came from "real-world" studies related to health and medicine. Among others he is the coauthor of the paper "Teledentistry: A Future Solution in the Diagnosis of Oral Lesions: Diagnostic Meta-Analysis and Systematic Review" that has already appeared online. (He is also co-author of three other papers of this sort but those we are not listing as they are not yet available online.)

Apart from the above a paper of Imre Csiszár and Thomas Breuer applying information theory for questions of finance was originally written earlier than the beginning of our project but its revision was taking place already during the project and the paper appeared in 2018. It concerns the minimization of the expectation of a random variable over a family of plausible prior distributions.

Information theory related combinatorics:

Dániel Soltész achieved several results on a problem area initiated in a 2012 paper by Körner, Messsuti and Simonyi [4]. The general problem considered here is to determine the maximum number of Hamiltonion paths one can give on the vertex set $[n] = \{1, \ldots, n\}$ such that the union of any two of them contains a specific subgraph, for example a cycle of prescribed length or a complete graph on 4 vertices. These kinds of problems are closely related to that of what is called the permutation capacity of graphs and constitutes a further extension of the problem area around Shannon's graph capacity. It is observed in [4] that when the requirement is that the union of any two Hamiltonian paths of our collection should contain an odd length cycle then the requested maximum is easy to determine and the authors asked whether the same maximum can be achieved if one insists that the required odd length cycle should be a triangle. Jointly with István Kovács Dániel Soltész gave a highly non-trivial construction proving that the answer to this question is affirmative. (This was achieved before the project started but after the application for it was submitted. The revision of the first version of the paper was already during the period of the project. The paper appeared in the Journal of Combinatorial Theory, Ser. B in 2018.) Soltész went on on this line of research and further developing their method introduced in the previously mentioned paper also obtained (jointly again with István Kovács) results when the required cycle has length k and it is obtained in a special way, namely one of the two Hamiltonian cycles involved contributes only one edge. (For a triangle this is always the case.) This paper appeared in the SIAM. J. Discrete Math. in 2019.

He also obtained results specifically when a given length even cycle should be obtained in the union of every pair of Hamiltonian paths in our family. While in case of odd cycles the requested maximum number of Hamiltonian paths is only exponential in n, for the even case it is larger for every fixed k (the length of the required cycle). In the paper "Even cycle creating paths" (J. Graph Theory, 2020) an upper bound by Cohen, Fachini and Körner [2] given for the $k = 4$ case is generalized for all even k. In joint work with Gergely Harcos he further improved the upper bound for even k (Combinatorica, 2020). In particular, they managed to close the superexponential gap in the $k = 4$ case given in the already mentioned result of [2].

Another paper of Soltész appeared in 2018 whose original submission dates back to October 2015 that is much before the project started, However, a six page long report was received only in 2017 asking for a fundamental rewriting. The paper was finally accepted in 2018. The paper gave new upper bounds for an old conjecture published in [1] concerning an extremal set theory type question which originated in a problem about efficient coding for memories.

Jointly with Péter Erdős, Tamás Mezei and István Miklós, Dániel Soltész also obtained results on rapidly mixing Markov chains that are defined on degree sequences of bipartite graphs. In another paper (joint with the above coauthors and also with Catherine Greenhill and Lajos Soukup) a unified approach was

established to prove rapid mixing to all so-called P-stable degree sequences. In a third paper (joint with Péter Erdős, Ervin Győri, Tamás Mezei and István Miklós) they also gave a non-trivial family of degree sequences that are not P-stable and the switch Markov chain is still rapidly mixing on the members of this family. (This latter paper appeared earlier than the previous one.)

Much effort was put to generalize an old result of Sali and Simonyi [9] in which they proved that a self-complementary graph and its complement can always be oriented in an isomorphic way so that the union of the two graphs is a transitive tournament. This had consequences on the relation between the Shannon capacity of a graph and the Sperner capacity of its oriented versions. The intended generalization was to show something analogous for three isomorphic graphs the edge sets of which partition the edge set of a complete graph. We found (jointly with Attila Sali and Gábor Tardos) that the analogous statement does not hold in general and found some conditions that make it happen. These conditions can be formulated for k-partitions in place of 3-partitions and for $k = 2$ they give back the original theorem. In this sense we managed to generalize it. However, we would have liked to completely characterize those cases when the required orientation is possible and it took quite some time to accept that this seems out of reach. First we have shown that the above mentioned sufficient conditions are not necessary by finding other sufficient conditions. The union of these, however, were still not necessary, so we had to be satisfied with giving some necessary but not sufficient conditions as well. These finding also helped determining the Sperner capacity of certain oriented graphs for which we did not know it before. The paper finally appeared in Order.

Several years ago the investigations of Sperner capacity led in [5] to the investigation of the parameter called local chromatic number that became especially interesting for Kneser and Schrijver graphs and other graphs whose chromatic number is determined by Lovász's topological method. We already had some joint results with Gábor Tardos on the structure of 4-chromatic Schrijver graphs that we started to work on again lately and managed to characterize the color-critical edges of 4-chromatic Schrijver graphs. We also found results on color-critical edges of general (not just 4-chromatic) Schrijver graphs (that are always vertex-color-critical, but generally not edge-color-critical). In particular, we proved that only so-called interlacing edges can be color-critical. In the 4-chromatic case this was already enough for a full characterization of color-critical edges, but the general case is more complicated.

More recenty, with Anna Gujgiczer we investigated the criticality behaviour of Schrijver graphs with respect to the fractional chromatic number. The latter is a well-known coloring parameter that also has an interesting information theoretic interpretation, namely it is the minimum zero-error capacity with feedback of those communication channels for which the given graph can serve as their distinguishability graph (cf. [8]). Schrijver graphs have the same fractional chromatic number as the Kneser graphs with the same defining parameters, but unlike for the chromatic number they are not critical for the fractional chromatic

number. In our paper, that is still under review, we located a vertex-critical subgraph with respect to the fractional chromatic number within Schrijver graphs.

Two years ago we spent a substantial amount of time (with Anna Gujgiczer and Gábor Tardos) on an interesting problem which is also related to the above topic, namely the chromatic behaviour of graphs whose chromatic number (and for certain parameters also its local chromatic number) is determined by the topological method. We managed to solve this probem but shortly after that also discovered that it was already known and published by other researchers earlier. It concerns an interesting property of the so-called generalized Mycielsky construction, namely that it does not always increase the chromatic number. It was observed that for complementary graphs of odd cycles it depends on the residue of the length modulo 4 whether the chromatic number should increase or not and that looked somewhat mysterious first. Finally we found the reason behind this phenomenon closely related to our earlier work with Gábor Tardos in [10] and even managed to generalize it from the complements of odd cycles to so-called rational complete graphs and then discovered that this is actually just a special case of a theorem by Zhishi Pan and Xuding Zhu [7]. So this did not become a result publishable in the usual way. However, this year we decided to write a short conference paper on the odd cycle complement case with the admitted goal of popularizing this nice special case of the result of Pan and Zhu that we felt not to be as well-known as it should be. This was presented at the 12th Japanese-Hungarian Symposium on Discrete Mathematics and Its Applications in Budapest this spring.

Anna Gujgiczer also worked on topics related to generalized Myczielskians of odd cycles, her paper (joint with Reza Naserasr, Rohini S. and S. Taruni) has recently been submitted. They give a simple proof of the known result that the generalized Myczielskian of an odd cycle has not only chromatic number 4 but even circular chromatic number 4 which is stronger. They also construct some special signed graphs (graphs where each edge comes with a positive or negative sign) satisfying some special requirements, including that their circular chromatic number (which has also been defined in the signed setting recently) is 4 and their negative girth (the length of the shortest cycle with an odd number of negative edges) is high.

In the paper "On colorful edge triples in edge-colored complete graphs" we gave a lower bound on the Shannon capacity of the Grötzsch graph which is the Myczielskian of the 5-cycle, showing that its Shannon capacity is strictly larger than that of the 5-cycle itself. (Recently we have found more general results of this type but that is still work in progress.) The paper contains several related Ramsey-type results as well.

Investigations of the behavior of the Shannon capacity of graphs under the so-called categorical product are initiated in one of our works. Any graph parameter that is nondecreasing via graph homomorphisms has the property that its value on this product is bounded from above by the individual values of the same parameter on the two graphs whose product is considered. It was conjec-

tured by Hedetniemi more than fifty years ago that this inequality is in fact an equality for the chromatic number. In these new investigations the analogous question is considered for the Shannon (OR-)capacity of graphs. There is little reason to believe that the equality holds here. Yet, using a recent and very strong result of Zuiddam [13], we showed that either it does hold, or if not then it also cannot hold for some much "better behaving" invariant as well. Only a little after the first conference version of this result has been submitted came the news that the half century old Hedetniemi conjecture has been disproved. (This made perhaps even more interesting the question of what other graph parameters may satisfy the analogous equality. It is, for example a celebrated result of Xuding Zhu [12] that the fractional chromatic number does.

The Hedetniemi conjecture had been refuted first only to graphs with very high chromatic number but by now it is pushed down to the minimal value where it fails (which is 5, for 4 it was known to hold for a long time). One step of this process was made by Claude Tardif in his paper [11] pushing down the current record to 14. He asked a question there about the multichromatic number of certain graphs that also came up in old work with Gábor Tardos [10] related to the local chromatic number of graphs having high chromatic number "for a topological reason". Tardif guessed an answer to his question that would have meant the nonexistence of some graph homomorphisms to certain Kneser graphs. If so, then it would have already pushed his current record lower as the lowest chromatic number for which Hedetniemi's conjecture fails. With Anna Gujgiczer we managed to prove that those homomorphisms actually exist. We also showed it more generally than what was required for Tardif's question. The paper containing this result appeared in the Journal of Graph Theory.

Joint results of Dániel Soltész with Ron Aharoni had their starting point also in the investigation of Kneser graphs and Schrijver graphs. In particular, they started to look at the question of how many vertex-disjoint Schrijver graphs may appear in the Kneser graph of the same parameters. This led to investigations of the function $f(n,k)$ that is the maximum number of Hamiltonian cycles on an n-element vertex set such that no two of these cycles share an independent set of size larger than k. They proved an interesting threshold phenomenon for this function, namely, that if $n = cn$ then there exists a threshold for c below which $f(n,k)$ is bounded by a constant while above it $f(n,k)$ is exponential in n. (This paper was submitted already after the application for this grant but before its start. Revisions were made even in 2018.)

A new type of coding problem was investigated by Anna Gujgiczer and the PI in joint work with Noga Alon, János Körner and Aleksa Milojecić. The codewords are $0-1$ sequences that can be considered as characteristic vectors of edge sets of graphs on a fixed vertex set. Thus one can consider the codewords as graphs and instead of the usual requirement of high code distance we can require certain structural properties of the graphs we obtain as the symmetric difference of any two "codeword-graphs". We looked at several such structural properties (for example, being connected, being Hamiltonian, containing a fixed graph as a subgraph, etc.) and gave exact results or upper and lower bounds

for the maximum possible size of a graph-code satisfying those requirements. The results show several surprising connections to classical results of graph theory. (This paper appeared in print in 2023 and already has three independent citations on arxiv.org.)

References

[1] Rudolf Ahlswede, Gábor Simonyi, Note on the optimal structure of recovering set pairs in lattices: the sandglass conjecture, *Discrete Math.*, 128 (1994), 389–394.

[2] Gérard Cohen, Emanuela Fachini, János Körner, Path separation by short cycles, *J. Graph Theory*, 85 (2017), 107–114.

[3] Imre Csiszár, Joint source-channel error exponent, *Problems of Control and Information Theory*, 9 (1999), no. 5, 315–323.

[4] János Körner, Sylvia Messuti, Gábor Simonyi, Families of graph-different Hamilton paths, *SIAM J. Discrete Math.*, 26 (2012), 321-329; arXiv:1106.3754 .

[5] János Körner, Concetta Pilotto, Gábor Simonyi, Local chromatic number and Sperner capacity, *J. Combin. Theory Ser. B*, 95 (2005), 101–117.

[6] Jie Luo, Anthony Ephremides, A new approach to random access: reliable communication and reliable collision detection, *IEEE Trans. Inform. Theory*, 58 (2012), 989–1002; arXiv:0909.0553 [cs.IT].

[7] Zishi Pan, Xuding Zhu, Multiple coloring of cone graphs, *SIAM J. Discrete Math.*, 24 (2010), 1515–1526.

[8] Claude E. Shannon, The zero-capacity of a noisy channel, *IRE Trans. Inform. Theory*, 2 (1956), 8–19.

[9] A. Sali, G. Simonyi, Orientations of self-complementary graphs and the relation of Sperner and Shannon capacities, *Europ. J. Combin.*, 20 (1999), 93–99.

[10] Gábor Simonyi, Gábor Tardos, Local chromatic number, Ky Fan's theorem, and circular colorings, *Combinatorica*, 26 (2006), 587–626, arXiv:math.CO/0407075.

[11] Claude Tardif, The chromatic number of the product of 14-chromatic graphs can be 13, *Combinatorica*, 42 (2022), 301–308,

[12] Xuding Zhu, The fractional version of Hedetniemi's conjecture is true, *European J. Combin.*, 32 (2011), 1168–1175.

[13] Jeroen Zuiddam, The asymptotic spectrum of graphs and the Shannon capacity, *Combinatorica*, 39 (2019), 1173–1184; arXiv:1807.00169 [math.CO].