Combinatorial Rigidity and its Applications

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This report gives a brief summary of the research results obtained within the framework of the research project Combinatorial Rigidity and its Applications, supported by the National Reseach, Development and Innovation Office - NKFIH, grant no. K-115483.

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1 Overview of the field and basic concepts

Rigidity and flexibility of structures is an exciting research area in the intersection of geometry, algebra, and combinatorics. Mathematicians have been interested in the rigidity of frameworks since Euler's conjecture from 1776, which stated that 3-dimensional polyhedra are rigid. The conjecture was verified for convex polyhedra by Cauchy in 1813 and for generic polyhedra by Gluck in 1975. Connelly constructed a counterexample to Euler's original conjecture in 1982. Interest and developments in rigidity have increased rapidly since the 1970's, motivated initially by the combinatorial characterization of rigid two-dimensional generic bar-and-joint frameworks by Laman in 1970 [18], and also by applications in many areas of science, engineering and design.

Combinatorial rigidity, the focus of this research project, refers to the part of rigidity theory which is concerned with those results and problems where the underlying combinatorial structure of the frameworks plays a key role. Maxwell pointed out, already in the 19th century, that one can deduce necessary conditions for the rigidity of a bar-and-joint framework by using properties of its underlying graph. Furthermore, the applications have encouraged mathematicians not only to develop theoretical results but also fast algorithms, e.g. for determining whether a given framework is rigid. These types of problems also made the combinatorial aspects (graph algorithms, combinatorial optimization) even more central. Results of this field are often useful in other areas of discrete geometry as well. This area has been extremely active in the last two decades and it is still expanding.

The main goal of this research project was to contribute to rigidity theory and its applications by new results in combinatorial rigidity, to publish the new results in top international journals and conference proceedings, and to continue the collaborations with leading researchers of the field from the USA, UK, and Japan. An additional goal was to bring new doctoral students to this area.

Basic concepts

The basic object of rigidity theory is the bar-and-joint framework, or simply framework. A (d-dimensional) *framework* is a pair (G, p), where $G = (V, E)$ is a graph and $p : V \to \mathbb{R}^d$ is a map. The length of an edge uv in (G, p) is the Euclidean distance between $p(u)$ and $p(v)$. We also call (G, p) a *realization* of G in $\mathbb{R}^d$. One may think about the framework as a collection of rigid (fixed length) bars connected at universal joints, corresponding to the edges and vertices, respectively. The bars can freely rotate about the joints provided that their length is unchanged.

The framework is said to be *globally rigid* in $\mathbb{R}^d$ if (G, p) is uniquely determined by its edge lengths up to congruence, that is, if all d-dimensional realizations of G with the same edge lengths as in (G, p) are congruent to (G, p). The framework is *rigid* in $\mathbb{R}^d$ if unicity holds in a small neighbourhood, that is, if there is an $\epsilon > 0$ such that all d-dimensional realizations (G, q) of G with the same edge lengths as in (G, p), and with $|p(v) - q(v)| < \epsilon$ for all $v \in V(G)$, are congruent to (G, p). It can be shown that (G, p) is rigid if and only if every continuous motion of the framework in $\mathbb{R}^d$ which preserves all the edge lengths takes the framework to a congruent framework, that is, it comes from a rigid motion (translation, rotation) of the space.

A framework (G, p) is called *generic* if there is no algebraic dependency between the coordinates of the points. A graph G is called *globally rigid* (resp. *rigid*) in $\mathbb{R}^d$ if all generic frameworks (G, p) on G are globally rigid (resp. rigid) in $\mathbb{R}^d$. It is known that for generic frameworks in $\mathbb{R}^d$ both global rigidity and rigidity depend only on the graph G (for every fixed dimension $d \geq 1$).

One can use frameworks to model the robustness or flexibility of several structures in which certain objects must preserve their distance but some others are free to move. This leads to a wide range of applications in engineering [20], molecular biology [22], sensor networks [1], CAD [19], deployable antennas [21], autonomous formations [2], and in other parts of discrete geometry.

2 New results

We start the survey of the new results by presenting the ones on global rigidity.

Global rigidity

It follows from the above mentioned old result of Cauchy that the graphs of triangulated convex polyhedra (which are the maximal planar graphs, or triangulations) are rigid in $\mathbb{R}^3$. Whiteley conjectured that if at least one new edge (a so-called bracing edge) is added to a triangulation so that the resulting graph is 4-connected then it becomes globally rigid in $\mathbb{R}^3$. Note that 4-connectivity is a necessary condition of three-dimensional global rigidity. With Shin-ichi Tanigawa we verified this conjecture by proving that a braced triangulation is globally rigid in $\mathbb{R}^3$ if and only if it is 4-connected [14]. The complete characterization of globally rigid graphs in $\mathbb{R}^3$ is still a major open problem.

We started a new research direction by obtaining the first theoretical results on the global rigidity of unit ball (in the plane: unit disk) frameworks and graphs. In such a framework two vertices are connected by an edge if and only if their distance is less than one (or equivalently, less than the so-called uniform sensing radius). Motivated by the applications in sensor network localization, it is natural to investigate those unit ball frameworks which are not globally rigid in the original sense, but within the family of unit ball frameworks they are the unique realizations of their edge lengths. We may call them unit ball globally rigid. We introduced these concepts and proved a number of results. Among others we showed that every saturated non-globally rigid unit ball graph in the plane has a unit ball globally rigid generic unit ball realization in $\mathbb{R}^2$, [3].

A recent result of Gortler, Thurston, and Theran [5] initiated another new research direction. They proved that if G is globally rigid in $\mathbb{R}^d$ then already the unlabeled set of edge lengths of a generic realization of G, together with the number of vertices, uniquely determine the graph as well as the realization (up to isomorphism and congruence, respectively). It holds for all $d \geq 2$. Here unlabeled refers to the fact that the graph structure is not given - just a set of real numbers, the edge lengths. With Dániel Garamvölgyi we investigated the case when the graph is not globally rigid [4]. Here we cannot expect the uniqueness of the realization, but in some cases we may be able to reconstruct the graph. We gave a complete solution in $\mathbb{R}^2$ in the case when G is rigid but not redundantly rigid. (We say that a graph G is redundantly rigid in $\mathbb{R}^d$ if $G - e$ is rigid in $\mathbb{R}^d$ for every edge e of G. It is known that globally rigid graphs on at least $d+2$ vertices are redundantly rigid in $\mathbb{R}^d$.) We also showed that if G is redundantly rigid but not globally rigid then a characterization is unlikely, since, in some sense, the difficulties are comparable to that of the graph isomorphism problem.

Recent work with Shin-ichi Tanigawa led to new results on globally rigid squares of graphs in $\mathbb{R}^3$ as well as on rigid and globally rigid graph powers in higher dimensions. The motivation comes from the fact that squares of graphs (also called molecular graphs) occur in the study of flexibility of molecules as well as in localization problems. We first formulated a conjecture (a necessary and sufficient condition for the global rigidity of squares) and then proved the necessity part. We also gave a connectivity based sufficient condition and proved some conjectures of Cheung and Whiteley on higher dimensional global rigidity of powers of graphs. We expect to obtain a few more results before finalizing this manuscript [15].

We published a conference paper on extremal problems concerning rigid and globally rigid graphs [8]. It contains a conjectured upper bound on the number of edges and the minimum degree of a minimally globally rigid graph in $\mathbb{R}^d$, which might give a useful tool in the analysis of globally rigid graphs. We also verified the conjectured bound in some special cases.

Further results on rigidity and global rigidity

Like in the case of global rigidity, the complete characterization of rigid graphs in $\mathbb{R}^d$, for $d \geq 3$, is still a major open problem. We studied the situation when the difference $p := n - d$ between the dimension and the number of vertices of the graph is small. We gave the excact characterization for rigidity and global rigidity in the case when p is at most four and pointed out that testing rigidity or global rigidity is fixed parameter tractable when parameterized by p [9]. In other words, there is an efficient algorithm for the problem for small (fixed) p.

A new type of problem that we studied is the characterization of those rigid frameworks (G, p) for which we have that in every other realization (G, q) of G with the same edge lengths, the distance between a pair of vertices u, v is at least as large as their distance in (G, p), for every vertex pair u, v. It turns out that this problem is non-trivial even in $\mathbb{R}^1$. We gave a solution in this special case in [17].

With Bill Jackson and Shin-ichi Tanigawa we continued the study of matrix completion problems. In these problems the goal is to decide whether a partially filled matrix has a unique completion to a positive semi-definite matrix of a given rank. This problem has a lot in common with rigidity and global rigidity and the tools from rigidity theory turned out to be useful. We provided new combinatorial sufficient conditions [6]. These results appeared as an invited paper in the special issue Fifty years of The Journal of Combinatorial Theory of the journal J. Comb. Theory (B).

Highly redundantly rigid or globally rigid graphs give rise to interesting extremal problems, motivated also by applications in formation control. For example, we say that a graph $G = (V, E)$ is k-edge-rigid (k-edge-globally rigid, respectively), if $G - F$ is rigid (globally rigid) for all $F \subset E$ with $|F| \leq k - 1$. In the last few months of the project we gave tight bounds for the minimum number of edges in a k-edge-rigid graph on n vertices in $\mathbb{R}^d$, for $d = 2, 3$, in terms of n and d. We also considered the removal of vertices as well as global rigidity. We plan to continue these investigations. The first results have been summarized in [10, 13],

Algorithms

A natural graph optimization problem is to search for a minimum cost globally rigid spanning subgraph in a graph with edge costs. In the metric version the graph is complete and the cost function on the edges satisfies the triangle inequality. The problem is NP-hard, even in the metric case. It contains the well-studied minimum cost 2-connected spanning subgraph problem as a special case ($d = 1$). We gave several approximation algorithms, including one which gives a 2-approximation in the two-dimensional metric version. We published these results as an invited book chapter of the Building Bridges II. conference (Budapest, 2018).

We studied the extension of the famous Steiner tree problem to count matroids. In this problem the input is a graph G, a vertex set T of terminals, a non-negative cost function w on the edges of G, and two integers k, l. The goal is to find a shortest (k, l)-tight subgraph of G which contains all the terminals. Tightness is defined by the sparsity parameters k, l: a subgraph H on n' vertices is (k, l)-tight if it has $kn' - l$ edges and no subset of n'' vertices of H spans more than $kn'' - l$ edges, for all $n'' \leq n'$. For $k = l = 1$ we obtain the trees, while for $k = 2$, $l = 3$ we get the minimally rigid graphs in two dimensions. It turned out that this problem has not been studied before, except for the Steiner tree problem ($k = l = 1$).

We have explored the complexity status of this problem and showed that it is NP-hard even if $k = 2$, $l = 3$, and either w is metric, or w is uniform and there are only two terminals. We designed approximation algorithms for several special cases, including the metric version of the shortest T-rigid subgraph problem. We also showed that the metric version can be solved in polynomial time for $k = 2$, $l = 3$, provided the cardinality of T is fixed [11].

Handbook chapters

The PI has been invited (with co-authors) to write two handbook chapters. The first one, with Walter Whiteley, appeared as one of the new chapters added to the third edition of the well-known Handbook of Discrete and Computational Geometry [16]. The other one, with Bill Jackson and Shin-ichi Tanigawa, appeared in the Handbook of Geometric Constraint Systems Principles. Both chapters are about globally rigid frameworks and graphs.

3 Conferences, lectures, doctoral students

We attended numerous conferences during the project and presented our results. Here we only list the invited talks: (i) LSE-QMUL Combinatorics Conference, London, UK, (ii) Bond-node structures, Lancaster University, UK, (iii) Building Bridges II., Budapest, Hungary, (iv) Graph Theory, Nyborg, Denmark, (v) Combinatorial optimization, RIMS, Kyoto, Japan, (vi) Rigidity of graphs, Lancaster University, UK, (vii) Circle packings and geometric rigidity, ICERM, Brown University, USA.

A postdoctoral researcher from Japan (Katie Clinch) spent one week in Budapest, supported by the grant. She gave a talk and took part in joint discussions.

Two doctoral students (Dániel Garamvölgyi and András Mihálykó) joined the ELTE Doctoral School in Mathematics, supervised by the PI.

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