Final report on the research project ”Degree bounds related to the rings of polynomial invariants of finite groups” (OTKA 113138)

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Let G be a finite group acting on a vector space V over a field $\mathbb{F}$. We denote by $\beta(G,V)$ the maximal degree of the elements generating the ring of polynomial invariant $\mathbb{F}[V]^G$. By Noether's classical result the supremum $\beta(G) := \sup \beta(G,V)$ is bounded from above by the group order $|G|$ (provided that the characteristic of $\mathbb{F}$ does not divide $|G|$). The main goal of our research was to find improvements of this result. I begin by summarising our main findings which were already published during this project.

In [1] we gave a reinterpretation in the framework of multiplicative ideal theory of the classical connection between the invariant theory of abelian groups and arithmetic combinatorics. This lead to a new proof of Nakajima's results on the class group of rings of polynomial invariants. We also found several analogies between the Davenport constant and the Noether number which are generalising some phenomena that were previously known only for abelian groups.

In [2] we have determined the precise value of the Noether number for every group of order less than 32. This work involved the development of a new method for giving lower bounds on the Noether number for those groups which have a representation as a reflection group. (This includes the important case of symmetric and alternating groups). The observations drawn from these examples have led us to formulate the following general conjectures: (1) the Noether number $\beta(G)$ does not depend on the the base field $\mathbb{F}$, (2) and it equals the Noether number $\beta(G,V)$ where V is the maximal multiplicity-free representation of G. We confirmed these conjectures for a large class of groups.

In the same article [2] we also described an algorithm for determining the small and large Davenport constants $\mathsf{d}(G)$ and $\mathsf{D}(G)$, and we determined their precise value for the same range of groups. These quantities were introduced and studied earlier by Geroldinger and Grynkiewicz in the hope that they could provide a lower and upper bound on $\beta(G)$. Our study however has uncovered a counterexample to this conjecture. We gave a full theoretical proof why this particular case constitutes a counterexample.

The main result of [3] states that for any finite p-group G the inequality $\beta(G) \geq \frac{1}{p}|G|$ holds if and only if G contains a cyclic subgroup of index p or it is one of two particular groups of small order. During the proof we have also established some upper bounds on the Noether number of the Heisenberg group (i.e. the extraspecial group of order p^3 and exponent p.) This constitutes a considerable advance towards the goal described in paragraph 2. of our research proposal to show that $\beta(G) \geq \frac{1}{p}|G| + p - 1$ holds if and only if G contains a cyclic subgroup of index p.

In [4] we have proved that the Noether number is monotone in the sense that for any proper subgroup $H \lneq G$ we must have $\beta(H) \lneq \beta(G)$. (Previously only the non-strict version of this inequality was known.) Even more, for the particular case when $N \triangleleft G$ is a normal subgroup we have obtained the stronger lower bound $\beta(G) \geq \beta(N) + \beta(G/N) - 1$. (Previously this was only known for the case when G/N is abelian).

Besides the results listed above we have also obtained some further results which are part of some still ongoing research that hopefully will be completed and published in the near future. With my coauthor I. Szőllösi we are still actively developing a new algorithm for obtaining the generators of the invariant rings which exploits some new Groebner-basis techniques and the knowledge of the automorphism group of the invariant ring.

As a byproduct of our study of the Davenport constants of arbitrary finite groups, we have extended an idea due to Alon, related to the so called rainbow matchings, thereby giving a short new proof of a classic result of Olson, which generalised the Erdos-Ginzburg-Ziv theorem. We have also defined an invariant theoretic analogue of the Erdos-Ginzburg-Ziv constant $\mathsf{E}(G)$ and have proved some generalisations of the related results in this invariant theoretic context.

The major focus of our research was directed lately towards the determination of the Noether number for the case of transitive permutation groups. Based on the study of several examples we have formulated the conjecture that for any group G acting transitively on a basis of a vector space V, the Noether number $\beta(G,V)$ is not greater than $\dim(V)$, which in many cases is much smaller than the formerly available upper bound $|G|$. We have already proved this conjecture for the case of all p-groups.

In the last period we have also made some advances towards determining the Noether number for affine groups (which also includes the case of Pawale's conjecture, which was one of the main motivations of our research).

Finally let me list all the conferences and seminars where I had the chance to participate and give talks on the above result during this time period:

1. Arithmetic and Ideal Theory of Rings and Semigroups, Graz, September 22-26, 2014
2. Groups and Rings, Theory and Applications, Sofia, July 15-22
3. Combinatorial and Additive Number Theory, Graz, January 2016
4. Representation Theory of Symmetric Groups and Related Topics, Kaiserslautern, February 2016
5. Algebraic Combinatorics and Group Actions, Herstmonceux Caste, July 2016
6. Technische Universitat, Munchen, Algebra Seminar, November 2017
7. Conference on Rings and Factorisations, Graz, October 2018
8. Babes-Bolyai University, Algebra Seminar, Cluj-Napoca, June 2018

References

[1] K. Cziszter, M. Domokos, A. Geroldinger *The interplay of invariant theory with multiplicative ideal theory and with arithmetic combinatorics* In: Scott Chapman, Marco Fontana, Alfred Geroldinger, Bruce Olberding (ed.) Multiplicative Ideal Theory and Factorization Theory: Commutative and Non-commutative Perspectives. (170) Berlin; Heidelberg; New York: Springer, 2016. pp. 43-95. (Springer Proceedings in Mathematics & Statistics; 170.)

[2] K. Cziszter, M. Domokos, I. Szőllösi, *The Noether numbers and the Davenport constants of the groups of order less than 32*, Journal of Algebra 510: pp. 513-541. (2018)

[3] K. Cziszter *The Noether number of p-groups* Journal of Algebra and its Applications (2018)

[4] K. Cziszter, M. Domokos, *Lower Bounds on the Noether Number*, Transformation Groups 23:(31) pp. 1-12. (2018)